Dealership or Marketplace with Fulfillment Services: A Dynamic Comparison

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Problem Definition: We consider two business models for a two-sided economy under uncertainty: dealership and marketplace with fulfillment services. Although both business models can bridge the gap between demand and supply, it is not clear which model is better for the firm or for the consumers. **Methodology/Results:** We show that while the two models differ substantially in pricing power, inventory risk, fee structure, and fulfillment time, both models share several important features, with the revenues earned by the firm from the two models converging when the markets are thick. We also show that for thick markets there is a one-to-one mapping between their corresponding optimal policies. **Managerial Implications:** Our results provide guidelines for firms entering two-sided markets: when the market is thick, the two business models are similar; when the market is thin, they should carefully inspect a number of market conditions before making the choice.

Key words: dynamic pricing, business models, dynamic game, Markov perfect equilibrium

1. Introduction

Intermediaries are often needed to facilitate transactions in two-sided markets, which consist of the supply side (sellers) and the demand side (buyers). There are two common business models for an intermediary in two-sided markets: dealership and marketplace. A dealer sets bid (ask) prices to buy from (sell to) the supply (demand) side of the market. The two sides of the market interact only indirectly through the dealer who runs an inventory. Dealerships have existed for a long time, especially in the automobile industry. For example, CarMax, a dealer company that buys and resells used cars, bridges consumers who want to buy used cars and car owners who want to sell used cars but do not want to deal with the hassle of direct sales. In a marketplace (e.g. retail platforms eBay, Amazon and Taobao, or ride-sharing platforms such as Uber and Didi), the firm provides a platform that connects the two sides directly. See, e.g., Parker et al. (2016) and Cusumano et al. (2019).

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Background. The two business models seem to be drastically different. For consumers, a dealer offers immediate fulfillment eliminating waiting costs, but the transaction might cost more relative to a marketplace; choosing the marketplace might be associated with a tiring process of looking for counterparties and negotiating prices. From a firm's perspective, becoming a dealer means exercising the full pricing power, managing its own inventory and bearing the inventory risk. The differences between the two models are summarized in Table 1.

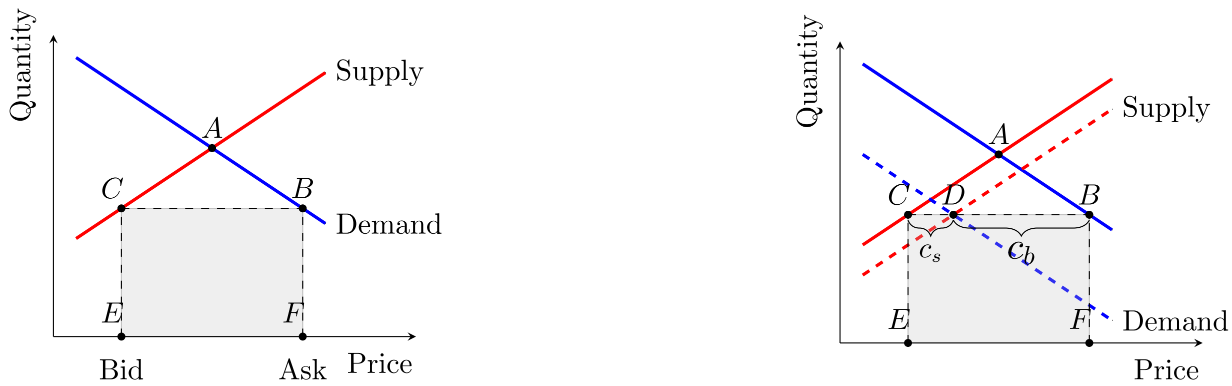
	Dealership	Marketplace
Pricing power	The dealer	Consumers
Inventory risk	Yes	No
Revenue source	Bid-ask spread	Transaction fees
Benefit for consumers	Immediate transaction	No middle man

Table 1 The differences between dealership and marketplace.

We next argue that the two models are essentially equivalent under a static, deterministic demand and supply model as illustrated in Figure 1. Point A in Figures (1a) and (1b) is the equilibrium at which demand and supply intersect in a frictionless market without an intermediary. For a dealer, with the bid and ask prices represented by E and F in Figure (1a), the resulting supply and demand are, respectively, B and C. To make a sustainable business, the prices should be set so that BC is horizontal balancing demand and supply. Thus, the earned bid-ask spread is represented by $|BC|$. In a marketplace (Figure 1b), demand and supply are shifted horizontally by the transaction fees charged from both sides, c_s for the sellers and c_b for the buyers. Thus point D is the equilibrium in the marketplace. The firm's earned fee is $c_s + c_b = |BC|$ per transaction. The associated quantity is the y value of D, or equivalently that of B or C. This demonstrates a one-to-one correspondence between the policies of the two models. Moreover, since the matched demand and supply in both business models are the y value of B or C, the total profit/revenue is represented by the area of the rectangle formed by $BCEF$. As a result, both models aim to maximize the same target in the static versions of the two models.

A natural question arises: if the revenues of the two business models are comparable under a static model, why is one particular business model preferred under certain market conditions? For example, according to a market survey of the luxury market,[1] the market share of second-hand luxury goods is well above 20% in developed countries such as Japan, U.S., and France. In those markets, consumers have seen the coexistence of dealership and marketplace in the secondary market. In developing countries like China, the market share is only 2% and the dealership model

[1] http://baogao.chinabaogao.com/baihuo/392626392626.html



(a) Dealership **(b)** Marketplace

Figure 1 The fundamental connection between the dealership and the marketplace.

is dominant. The thickness of the market seems to play a role. As another example, Gautier et al. (2023) empirically find that Amazon is more likely to sell products as a middleman (dealer) when the market size of the demand side is relatively large. The market composition of buyers and sellers seems to affect the adoption of business models. Third, haoche51, a Chinese online platform for used cars, started in 2014 as a marketplace to match sellers and buyers directly, and later transformed into a model that is more similar to dealership in 2016. The CEO explained why the first business model failed in an interview[2] stating that many buyers and sellers are unwilling to wait, and the new C2B2C (dealership) model better accommodates their needs. The impatience of consumers waiting for a matching also factors in. The static model represented by Figure 1 is unable to capture the factors that make one business model better than the other.

For this reason, it is important to extend the analysis to the realm of stochastic dynamic models. It is much closer to the market environment faced by the intermediaries: the supply and demand are random and fluctuate over time due to intertemporal effects such as inventory risk in the dealership model and sellers waiting for buyers in the marketplace. Our goal in this paper is to study stochastic dynamic models that incorporate inventory risk and waiting times to shed light on the differences between the two business models. Following the large literature after Rochet and Tirole (2003, 2006), we focus on the two-sided non-financial marketplaces that set a high exit cost for sellers, common for platforms with fulfillment services on which sellers cannot retrieve their products once they join the market. For instance, ThredUp[3] is one of the major luxury resale platforms. Once sellers join the platform, ThredUp requests them to ship their products to the inspection center through its fulfillment service. Then each authenticated product will be listed on the website, and its owner can dynamically adjust the price until it is sold. Once listed on the

[2] https://m.qctt.cn/news/120581

[3] https://gap.thredup.com/pages/seller-terms

platform, the product cannot be withdrawn within a time window (30 days or 45 days) and the withdrawal cost is $5.99 service fee plus the shipping cost. When a transaction occurs, the platform delivers the product to the seller and transfers the payment minus the commission to the seller. Such a business model, sometimes referred to as "consignment", has been adopted by an increasing number of resale platforms (e.g., Lampoo, Luxury Closet, Zhuanzhuan).

Our Contribution. In this paper, we build dynamic models for the dealership and the marketplace as monopolies. We first study a dealer that sets dynamic ask/bid prices based on the on-hand inventory to maximize the long-run discounted profit, when the supply and demand arrive as Poisson processes. We then study a marketplace platform that determines optimal transaction fees. Sellers set prices dynamically based on the transient supply/demand imbalance and their waiting costs in a dynamic game among themselves; we characterize the Markov perfect equilibrium of the dynamic game. The contribution of this paper is threefold:

(1) We build *dynamic* models under the common market environment for both two-sided markets to capture the demand and supply fluctuations and the intertemporal effects. This contributes to the literature which largely focuses on static equilibrium models.

(2) We derive the optimal policies of the two models and explore their structural properties. Despite the differences summarized in Table 1, the optimal policies share several salient features: (i) The firms always benefit from higher inventory levels or more sellers but the marginal value is diminishing. (ii) More inventory or sellers both lead to a lower transaction price, though in the dealership market the transaction price is determined by the monopolistic dealer and in the marketplace it is an equilibrium outcome of the price war among sellers. (iii) The quantities measuring *transient supply relative to demand* exhibit "mean-reversion", demonstrating the self-regulatory nature of the two models. For the dealer, it is the inventory level; in the marketplace, it is measured by the number of sellers waiting to be matched.

(3) We show that either model can be more profitable depending on the market conditions. In particular, if the sellers are impatient, or the market size of buyers is relatively small, then the dealership model is more profitable, due to the high waiting cost of the sellers in the marketplace. It may explain why many online retailers like Amazon started with a dealership model at their early stage when the customer base has not been fully built. When both sides of the market become thicker, a fundamental connection between the two models emerges. The optimal bid/ask prices of the dealer and the optimal commissions charged by the platform converge to the quantities illustrated in Figure 1, confirming the intuition (the rigorous notion of convergence will be specified later). As a result, the optimal revenues of the two business models converge to the same value, when both sides of the market become thicker.

Literature Review. First, in terms of game theoretical models, there are papers studying two-sided markets by building static models to investigate governance structures, platform competitions (Rochet and Tirole 2003), platform investment (Njoroge et al. 2014), quality selection (Light et al. 2024), etc. Gautier et al. (2023) construct search models to compare two business modes, focusing on the impact of the market composition. Both Rust and Hall (2003) and Li et al. (2019) consider the competition among dealers. Li et al. (2019) show how the dealer's pricing and order decisions are affected by its inventory. We further illustrate how its decisions are affected by the market thickness and provide an asymptotic optimal pricing policy. There are also dynamic models studying the matching and bargaining of two-sided markets, such as Mortensen and Wright (2002), Duffie et al. (2005), Satterthwaite and Shneyerov (2007). Compared to this line of literature, our dynamic model focuses on the decisions of a dealer and a platform to maximize revenues, and we consider buyers and sellers of heterogeneous valuations; indeed, our model converges to an equilibrium associated with an optimization problem.

Second, our dealership model is related to the literature on dynamic pricing (Gallego and van Ryzin 1994). Our model is related to an inventory system with joint inventory and pricing decisions under stochastic demand (Chen and Simchi-Levi 2004a,b, Chen and Gallego 2019). The main difference in our model is that the inventory is not replenished by ordering a certain quantity from a manufacturer; rather, it is replenished by setting a dynamic price to buy from individual sellers, who arrive in a stochastic fashion. Such symmetry between the demand and supply sides of the inventory gives rise to the bid-ask spread and other unique structural properties in our model. Chen and Hu (2020) study the strategic behavior of buyers and sellers in the platform and show a fixed pricing policy is asymptotically optimal. In a closely related recent paper, Ghuloum et al. (2022) study the dealership model and develop a similar set of structural results. We summarize the differences in Table 2. More importantly, we focus on the analysis of the marketplace model in this study.

	Ghuloum et al. (2022)	Our work
Time horizon	Discrete and finite	Continuous and infinite
Arrival rates	Time-inhomogeneous	Time-homogeneous
Comparative statics	Inventory level	Inventory level, discount factor and arrival rates
Optimal margin	Lower bound	Numerical experiments
The fixed-price policy	Numerical experiments	Theoretical analysis

Table 2 Comparison of the dealership model between Ghuloum et al. (2022) and our work.

Third, there are recent papers studying platforms in the sharing economy. Cohen and Zhang (2022) use a discrete choice model to study two-sided platforms' competition and coordination.

Birge et al. (2021) introduce a model to study a platform's optimal commission and subscription policy in a networked marketplace, where not all types of buyers and sellers are compatible. Taylor (2018) and Bai et al. (2019) use queueing models to examine the operations of on-demand platforms. Bimpikis et al. (2019) and Besbes et al. (2021) introduce the spatial or geographical elements into the pricing and matching of the platforms. The optimization problem related to Figure 1 is studied in Hu and Zhou (2017), which study a platform's optimal policy to set price, wage and commission; their focus is on the equilibrium state (without intertemporal fluctuations in demand and supply), which, interestingly, can be viewed as special cases of the limit of our models in a thick market. Benjaafar et al. (2022) consider an on-demand service platform with riders sensitive to the waiting time, strategic drivers and a profit-seeking intermediary. Although with a different model, their observation that a larger size of the opposite side does not necessarily increase the welfare coincides with ours. He and Goh (2022) use a dynamic model to capture the demand and supply fluctuations of a service platform, which can allocate the orders to its own employees or outsource them to freelancers. The objective is to adjust the order allocation for the optimal growth of the platform. Lian and Van Ryzin (2021) study the market growth decision of a two-sided platform where the reward for sellers and the price for buyers are determined by the platform, and the number of transactions is contingent on the sizes of both the demand and supply participant pools. Hagiu and Wright (2024) establish a marketplace model with one seller to study the leakage problem, i.e., customers may take transactions off the marketplace to avoid transaction fees.

Compared to this line of literature, our work differs in the following two regards. First, we focus on the connection and comparison of the two business models. To the best of our knowledge, no previous studies have analyzed the pricing/commission policies of the two business models in a *dynamic* setting and discovered their fundamental connection. Second, from the modeling perspective, our framework provides a novel approach to capture the trade-off between monetary and time values of sellers on a platform. Most previous papers resort to queueing models, which usually only allow for steady-state analysis and cannot handle transient strategic behaviors. Our framework leverages the competitive nature of the agents and links the waiting cost to the state of the dynamic game. It allows for tractable analysis of the transient dynamics of the game.

2. Two Dynamic Models for Two-sided Markets

Consider two crowds of consumers, buyers and sellers, for a single type of product. Consumers arrive according to two independent Poisson processes with rates λ_b and λ_s, respectively, for buyers and sellers. The random private valuations of the buyers (sellers) associated with one unit of the product are denoted by Ω_b (Ω_s). Therefore, for a random buyer (seller), charged (offered) a price p, the expected demand (supply) is given by $d_b(p) \triangleq \mathsf{P}(\Omega_b \geq p)$ and $d_s(p) \triangleq \mathsf{P}(\Omega_s \leq p)$. Next, we introduce the two dynamic models separately.

2.1. The Dealership Market

In the dealership market, the buyers and sellers cannot trade directly, but transact with each other indirectly through the dealer. In particular, the dealer sets the ask and bid prices dynamically. Buyers with valuations above the ask price (p_b) would buy from the dealer, whereas sellers with valuations below the bid price (p_s) would sell to the dealer. We assume that consumers make decisions upon arrival, and do not strategically wait for a different price. Thus, the arrivals of buyers and sellers who transact with the dealer follow two time-varying Poisson processes, $N_b(t)$ and $N_s(t)$, with rate $\lambda_b d_b(p_b(t))$ and $\lambda_s d_s(p_s(t))$, respectively, where $p_b(t)$ and $p_s(t)$ are the ask and the bid prices at time $t \geq 0$.

To simplify notations, we respectively define $\mathcal{R}_b(p,z) \triangleq d_b(p)(p-z)$ as the expected profit for the seller at ask price p, when his unit cost is z, and $\mathcal{R}_b(z) \triangleq \sup_{p\geq z} \mathcal{R}_b(p,z)$ as the optimal expected profit at a unit cost z. Similarly, we respectively define $\mathcal{R}_s(p,z) \triangleq d_s(p)(z-p)$ and $\mathcal{R}_s(z) \triangleq \sup_{0\leq p\leq z} \mathcal{R}_s(p,z)$ as the expected surplus for a buyer at bid price p, when his valuation is z, and the optimal expected surplus at valuation z. The assumption below is common in the pricing literature and we provide justifications in Section EC.1.

Assumption 1. *(1) The market environment is stationary, i.e., $d_b(\cdot)$ and $d_s(\cdot)$ do not change over time. (2) $d_b(p)$ ($d_s(p)$) is continuous and decreasing (increasing) for $p \in [0,\infty)$, with $d_b(\infty) = d_s(0) = 0$. When $d_b(p), d_s(p) \in (0,1)$, they are strictly monotone. (3) For any z, both $\mathcal{R}_b(p,z)$ and $\mathcal{R}_s(p,z)$ are unimodal and have a finite maximizer in $p \geq 0$.*

Formulation. At time t, the dealer uses the bid and ask prices $p_b(t)$ and $p_s(t)$ to maximize the expected revenue over an infinite horizon discounted at rate r, which can be expressed as:

$$V_{\mathcal{D}}(x) \triangleq \max_{p_b(t),p_s(t)} \mathsf{E}\left[\int_0^\infty e^{-rt}\big(p_b(t)dN_b(t) - p_s(t)dN_s(t)\big)\right]$$
$$\text{subject to} \quad x - N_b(t) + N_s(t) \geq 0, \quad \forall t \geq 0,$$

where x is the initial inventory level and the constraint means that the dealer can only sell if its inventory is positive. Let $\Delta V_{\mathcal{D}}(x) \triangleq V_{\mathcal{D}}(x) - V_{\mathcal{D}}(x-1)$ represent the marginal value of the dealer's inventory when he is holding x units. It is easy to show that $V_{\mathcal{D}}(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rV_{\mathcal{D}}(x) &= \lambda_b \mathcal{R}_b(\Delta V_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\Delta V_{\mathcal{D}}(x+1)) \quad x \geq 1, \\ rV_{\mathcal{D}}(0) &= \lambda_s \mathcal{R}_s(\Delta V_{\mathcal{D}}(1)) \end{aligned} \tag{1}$$

For any $z > 0$, we define $P_b(z) \triangleq \arg\max_{p\geq 0} \mathcal{R}_b(p,z)$ and $P_s(z) \triangleq \arg\max_{p\geq 0} \mathcal{R}_s(p,z)$. Then, from (1), the optimal bid and ask prices are functions of x, denoted as $p_s^*(x)$ and $p_b^*(x)$:

$$p_s^*(x) = P_s(\Delta V_{\mathcal{D}}(x+1)),\ x \geq 0; \quad p_b^*(x) = \begin{cases} P_b(\Delta V_{\mathcal{D}}(x)), & \text{if } x \geq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The *reflective* boundary condition in (1) at $x=0$ highlights the stockout risk faced by the dealer. The dealer tries to avoid having zero inventory, as potential arrivals of demand (and thus revenue) would be missed if $x=0$. This may nevertheless happen due to randomness; in this case, the inventory is "reflected" to $x=1$ at the next transaction due to a purchase by the dealer. Moreover, there is a (financial) *inventory risk* due to the discounted inventory value during the holding period (see Remark 2). Specifically, the inventory risk can be measured by the difference between the present value of the inventory at the buying time and the selling time, i.e., $\mathsf{E}\left[\int_0^\infty e^{-rt}p_b(t)dN_b(t) - e^{-rt}p_b(t)dN_s(t)\right]$. Note that the structural results in Proposition 1 are also derived in a different setting in Ghuloum et al. (2022). We provide the intuition and implications of the results in Section EC.1.

Proposition 1. *Under the optimal pricing policy, we have the following results. (1) The value function $V_{\mathcal{D}}(x)$ is increasing concave in x. (2) The existence of bid-ask spread: $p_b^*(x) \geq \Delta V_{\mathcal{D}}(x) \geq \Delta V_{\mathcal{D}}(x+1) \geq p_s^*(x)$ for all $x \geq 0$. (3) Monotonicity of bid and ask prices: (a) $p_s^*(x)$ and $p_b^*(x)$ are decreasing in x. Besides, $\lim_{x\to\infty} p_s^*(x) = 0$ and $\lim_{x\to\infty} p_b^*(x) = \arg\max_p\{d_b(p)p\}$. (b) $p_s^*(x)$ and $p_b^*(x)$ decrease if r increases or λ_b and λ_s decrease in proportion. (4) Mean-reversion of inventory: There exists $0 \leq x_p < +\infty$, such that when $x > x_p$, $\lambda_b d_b(p_b^*(x)) > \lambda_s d_s(p_s^*(x))$; when $x < x_p$, $\lambda_b d_b(p_b^*(x)) < \lambda_s d_s(p_s^*(x))$. Moreover, x_p increases if r decreases or λ_b and λ_s increase in proportion.*

Thick Markets and Fixed-Price Policies. Frequently the market sizes are large in the sense that the arrival rates of both buyers and sellers result in low coefficients of variation for both buyers and sellers (which are of the order of $1/\sqrt{\lambda_b}$ and $1/\sqrt{\lambda_s}$). This is the case for popular product categories. We study this "thick-market" regime and show that the fixed-pricing rule corresponding to the fluid approximation where demands are replaced by their expectations has desirable properties.

We consider the fluid approximation where the Poisson processes, $N_b(t)$ and $N_s(t)$, are replaced by their means. More precisely, with ask price p_b and bid price p_s, in the next time interval of dt, the dealer sells $\lambda_b d_b(p_b)dt$ units and buys $\lambda_s d_s(p_s)dt$ units exactly. (Hence the product is infinitely divisible and the inventory is a continuous variable.) Such fluid approximation is used widely in dynamic pricing as it sheds light on the first-order effects. Denote $\bar{V}_{\mathcal{D}}(x)$ as the optimal discounted revenue for the fluid approximation. The dealer attempts to maximize the discounted cash flow

$$\bar{V}_{\mathcal{D}}(x) = \max_{p_b(t),p_s(t)} \int_0^\infty e^{-rt}\big(\lambda_b p_b(t) d_b(p_b(t)) - \lambda_s p_s(t) d_s(p_s(t))\big)dt$$
$$\text{subject to} \int_0^t \big(\lambda_b d_b(p_b(s)) - \lambda_s d_s(p_s(s))\big)ds \leq x, \quad \forall t \geq 0. \tag{3}$$

Replacing $\Delta V_{\mathcal{D}}$ by the derivative $\bar{V}'_{\mathcal{D}}(x)$ in (1), we obtain the following HJB equation for $\bar{V}_{\mathcal{D}}(x)$:

$$r\bar{V}_{\mathcal{D}}(x) = \lambda_b \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x)), \quad x > 0. \tag{4}$$

Next, we need to find the boundary condition that exhibits an entirely different pattern from that of $V_{\mathcal{D}}(0)$. In the fluid approximation, by the continuity of $d_b(\cdot)$ and $d_s(\cdot)$ and the infinitesimal deterministic demand/supply flow, the inventory will always reach an equilibrium level $\bar{x}_p$ under the optimal pricing policy, such that $\lambda_b d_b(\bar{p}_b^*(\bar{x}_p)) = \lambda_s d_s(\bar{p}_s^*(\bar{x}_p))$, and not deviate from the equilibrium afterwards. Without uncertainty, the equilibrium $\bar{x}_p$ must equal zero, since stocking a positive inventory is meaningless after reaching the equilibrium, as it could have been turned into profit. Therefore, $\bar{V}_{\mathcal{D}}(0)$ should satisfy the following *absorbing* (instead of *reflective*) boundary condition:

$$r\bar{V}_{\mathcal{D}}(0) = \max_{\lambda_b d_b(p_b) = \lambda_s d_s(p_s)} \left\{ \lambda_b p_b d_b(p_b) - \lambda_s p_s d_s(p_s) \right\}. \tag{5}$$

After the inventory reaches zero, it stays in the absorbing state $x = 0$. In other words, the preferred inventory level is zero in the fluid approximation. This is significantly different from the stochastic formulation, which has a positive preferred inventory x_p to which the inventory process reverts. Moreover, we have shown that x_p is increasing in the scale or market size of the system (λ_s or λ_b), as illustrated in Figure 4c. The discrepancy is caused by the fact that although x_p is increasing in λ, x_p/λ is decreasing in λ and eventually converges to 0 in the fluid limit. It can also be shown that $\bar{V}_{\mathcal{D}}(x)$ is increasing and concave, and serves as an upper bound of $V_{\mathcal{D}}(x)$ (see Proposition EC.1 in Section EC.2 in the online supplement). In other words, eliminating uncertainty and stockout risk increases the dealer's profit. Moreover, the ratio $\bar{V}_{\mathcal{D}}(x)/V_{\mathcal{D}}(x)$ tends to one as $x \to \infty$.

A Fixed-Pricing Policy for the Fluid Model. In the revenue management literature (see, e.g., Gallego and van Ryzin (1994)), it is found that the fluid approximation and the associated fixed pricing heuristic work well in a "scaled" system. Here we consider a similar setting of the thick-market regime, in which we scale demand and supply simultaneously by n, i.e., $(\lambda_b^{(n)}, \lambda_s^{(n)}) = n(\lambda_b, \lambda_s)$.

PROPOSITION 2. *Consider a fixed-pricing policy $\{\hat{p}_b, \hat{p}_s\}$ that solves the absorbing boundary condition in (5), namely*

$$\max_{p_s, p_b} \quad \frac{1}{r}\lambda_s d_s(p_s)(p_b - p_s), \text{ subject to } \lambda_s d_s(p_s) = \lambda_b d_b(p_b), \tag{6}$$

and let $\tilde{V}_{\mathcal{D}}^{(n)}(x)$ denote the expected discounted revenue of the dealer using the fixed price policy $\{\hat{p}_b, \hat{p}_s\}$. Then for any $x \in \mathbb{N}$, we have

$$\frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{V_{\mathcal{D}}^{(n)}(x)} \geq \frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{\bar{V}_{\mathcal{D}}^{(n)}(x)} = 1 - O(\frac{1}{\sqrt{n}}) \text{ and } \lim_{n\to\infty} \frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{n} = \frac{1}{r}\lambda_s d_s(\hat{p}_s)(\hat{p}_b - \hat{p}_s).$$

Due to the dynamic effect of randomness, the relative loss of profit under the heuristic will be $O(1/\sqrt{n})$, yielding that the fixed-price heuristic is asymptotically optimal. Intuitively, this holds because, when the market becomes thick, the initial inventory is negligible compared to the mass stream of supply and demand from consumers.

2.2. The Marketplace

In a marketplace, the intermediate firm provides a platform that facilitates the *direct* transaction between buyers and sellers, instead of serving as a middleman, and usually charges fees per transaction from both sides. While sellers can potentially make more profits from transactions, they have to wait for counterparties, instead of immediately transacting with a dealer. The buyers may also get better prices because of competition among sellers. In this section, we introduce a dynamic continuous-time framework to capture those trade-offs and analyze the optimal fees charged by the platform (firm).

2.2.1. Basic Settings Suppose the platform charges sellers (buyers) c_s (c_b) per transaction.[4] We assume that c_s and c_b are set at time zero and fixed over time, because in practice transaction fees are usually stated upfront before consumers come to the platform and kept unchanged for a long time. For example, most platforms such as eBay inform the clients about the fee policy when they register. In the marketplace, there are sellers posting prices for their homogeneous goods. Both sides are assumed to have one unit of product to sell or buy. We use "he" to refer to the platform and "she" to refer to a seller or a buyer.

The Buyers. Buyers arrive at the marketplace at a Poisson process with rate λ_b. Their valuations are drawn from Ω_b independently whose tail distribution is $d_b(\cdot)$, as explained in Section 2.1. When a buyer arrives, she observes the sellers' quoted prices and purchases from the seller who offers the lowest price, provided that this price plus the transaction fee c_b is lower than her valuation of the product. If two or more sellers quote the same lowest price, then an arbitrary tie-breaking rule is introduced (e.g. random selection). Finally, a buyer who finds the lowest price plus c_b higher than her valuation leaves the marketplace permanently without purchasing — that is, she does not wait, which is consistent with the behavior in the dealership model. Consequently, when the lowest price is p, the random time elapsed before a buyer arrives and purchases a product follows an exponential distribution with rate $\lambda_b d_b(p+c_b)$, in which case a transaction occurs and the number of sellers in the marketplace decreases by one.

The Sellers. Sellers arrive in a Poisson process with rate λ_s, whose valuations have i.i.d. distributions with CDF $d_s(\cdot)$. An arriving seller observes m, the number of sellers, and their posted prices. Similar to the dealership model, a seller makes a take-it-or-leave-it decision upon her arrival

[4] In Section 5.1, we analyze fees proportional to the transaction price. Most results in this section hold.

at the platform: joining the platform and posting a price, or leaving the market permanently if the expected utility of joining the platform is less than her valuation. For the sellers, the expected utility of joining the marketplace is

$$f_m = \mathbb{E}[\tilde{p} - c_s - \kappa\tau], \tag{7}$$

where $\tilde{p}$ is the (random) future transaction price, τ is the (random) time elapsed between joining the platform and the time her product is sold; both $\tilde{p}$ and τ will be determined endogenously later. Compared to the platform, sellers typically care more about short-term returns. Therefore, we use linear waiting costs to characterize their time disutility instead of discounting, with a unit cost κ. If the seller decides to join, then the number of sellers in the marketplace is increased by one. Note that we mainly focus on platforms with fulfillment services that normally set a high exit cost for sellers. Therefore, once joining, the seller does not withdraw and leaves the marketplace only after her product is sold. The sellers in the marketplace can adjust their posted prices at any time without a cost, a practice adopted by many platforms such as Craigslist.

Remark 1. In this model, we make the assumption that once sellers enter the market, they remain active and do not quit, possibly due to the high exit cost. For many marketplace platforms, before sellers post their products, they need to ship their products to the marketplace by fulfillment services, such as Fulfillment by Amazon or Shopify Fulfillment Network. For example, ThredUp requires sellers to ship the products to its warehouse for authentication examinations before they are posted. Sellers cannot withdraw their products until the last week of the listing window (30 days or 45 days), and the reclaim cost is \$5.99 service fee plus the shipping cost.[5] As a result, in our model, the WTS only plays a role when the seller decides whether to join the marketplace but not after the entrance.

The Dynamics. Since an arriving buyer only transacts with the seller offering the lowest price, the sellers continuously undercut the current lowest price, leading to a price war. The price war ends immediately. As a result all m sellers are indifferent between missing the next transaction and selling the product at the lowest price. At the moment of the next transaction, the seller offering the lowest price, denoted as p_m, receives $p_m - c_s$ and leaves the market, while the future expected utility of the remaining $m-1$ sellers becomes f_{m-1}. By the discussion above, we must have $p_m - c_s = f_{m-1}$ for $m \geq 2$. For $m = 1$, the single seller does not face any competition and maximizes her utility by setting the optimal price p_1.

To understand the dynamics of the marketplace, consider the events that could happen in the next infinitesimal time period of length δt. For $m > 1$, the transition to $m-1$ is caused by a

[5] https://gap.thredup.com/pages/seller-terms

buyer arriving at the marketplace whose valuation is higher than the lowest price p_m plus c_b. In the following, we will refer to $p_m + c_b$ as the *payment* made by buyers when there are m sellers in the marketplace. Therefore, the transition probability is approximately $\lambda_b d_b(p_m + c_b)\delta t$. After the transition, the seller who has sold her item or not receives payoff $p_m - c_s$ or f_{m-1}, which are equal. On the other hand, the transition to $m+1$ is caused by an arriving seller, provided that her expected utility of joining the marketplace is higher than her valuation. Note that her expected utility is equal to the expected future utility of the $m+1$ sellers on the platform after she joins, which is f_{m+1}. Therefore, the transition probability is approximately $\lambda_s d_s(f_{m+1})\delta t$. For convenience, let $f_0 = p_1 - c_s$ and $w = c_s + c_b$.

Thus, given the number of sellers m at any time, the dynamics of the marketplace can be modeled as a continuous-time Markov chain with state m, i.e., a state-dependent birth-death process. More precisely, consider a time point when there are m sellers on the platform. They engage in an immediate *price war*, leading to a competitive price $p_m = c_s + f_{m-1}$. If a buyer arrives and her valuation is above $p_m + c_b$, which occurs with rate $\lambda_b d_b(p_m + c_b) = \lambda_b d_b(f_{m-1} + w)$, then a transaction happens and m becomes $m-1$. If a seller arrives and her valuation is below f_{m+1}, which occurs with rate $\lambda_s d_s(f_{m+1})$, then m becomes $m+1$. Thus the utility function f_m will satisfy the following recursive equations:[6]

$$
\begin{aligned}
f_1 &= \max_{f_0} \left\{ \frac{\lambda_b d_b(f_0 + w) f_0 + \lambda_s d_s(f_2) f_2}{\lambda_b d_b(f_0 + w) + \lambda_s d_s(f_2)} - \frac{\kappa}{\lambda_b d_b(f_0 + w) + \lambda_s d_s(f_2)} \right\}, \\
f_m &= \frac{\lambda_b d_b(f_{m-1} + w) f_{m-1} + \lambda_s d_s(f_{m+1}) f_{m+1}}{\lambda_b d_b(f_{m-1} + w) + \lambda_s d_s(f_{m+1})} - \frac{\kappa}{\lambda_b d_b(f_{m-1} + w) + \lambda_s d_s(f_{m+1})}, \quad m \geq 2,
\end{aligned} \tag{8}
$$

where the first term of the right hand is the expected payoff at the next transition, and the second term is the expected waiting cost before the transition.

Although the recursive system for f_m is derived for all $m > 0$, it does not necessarily mean that there will be infinite sellers in the marketplace. According to Assumption 1, the lower bound of the support of Ω_s is zero. Define $M = \min\{m : f_{m+1} \leq 0\}$. When there are M sellers in the marketplace, the expected utility of joining for an arriving seller is $f_{M+1} \leq 0$, which is always less than her valuation of the product. Therefore, no more sellers are willing to join the marketplace and the number of sellers in the marketplace never exceeds M. Because M is endogenously determined by solving the recursive system in (8), it is referred to as the *endogenous capacity* of the platform. We impose the following assumption.

ASSUMPTION 2. *There exists a finite price $\bar{p}$ such that $d_b(\bar{p}) = 0$.*

[6] Observe that only the sum $w = c_s + c_b$ matters here. Intuitively this is because, as long as w is fixed, changing c_s and c_b only shifts the demand and supply curves horizontally without changing other quantities.

Assumption 2 implies that the buyers' reserve price is bounded above. Combined with Assumption 1, we can show that the number of sellers is bounded above.

LEMMA 1. *Under Assumptions 1 and 2, the endogenous capacity M is finite, i.e., $M \leq 2\bar{p}\lambda_b/\kappa$.*

Platform's Objective. Let $V_{\mathcal{M}}(m)$ be the platform's expected discounted revenue when there are m sellers in the marketplace, at discount rate r. That is,

$$V_{\mathcal{M}}(m) = \int_{t=0}^{\infty} e^{-rt}(c_s + c_b)dN_t, \quad \text{given } m \text{ sellers in the marketplace at } t = 0,$$

where N_t is the counting process of the number of transactions up to time t given the *fixed* transaction fees c_s and c_b. Note that, since buyers and sellers transact directly, we have $N_b(t) = N_s(t)$ and there is no inventory risk for the marketplace. Similar to the derivation of f_m, we have

$$\begin{aligned} rV_{\mathcal{M}}(0) &= \lambda_s d_s(f_1)\Delta V_{\mathcal{M}}(1), \\ rV_{\mathcal{M}}(m) &= \lambda_s d_s(f_{m+1})\Delta V_{\mathcal{M}}(m+1) + \lambda_b d_b(f_{m-1}+w)(w - \Delta V_{\mathcal{M}}(m)), \quad 1 \leq m \leq M-1, \\ rV_{\mathcal{M}}(M) &= \lambda_b d_b(f_{M-1}+w)(w - \Delta V_{\mathcal{M}}(M)), \end{aligned} \tag{9}$$

where $\Delta V_{\mathcal{M}}(m) = V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ is the marginal value of an additional seller and $w = c_s + c_b$. The platform's objective is to choose c_s and c_b to maximize $V^*_{\mathcal{M}}(m) \triangleq \max_{c_s,c_b} V_{\mathcal{M}}(m)$.

Equilibrium Analysis. Given m, the value of f_m solves (8) and the associated Markov process follows a state-dependent birth-death process. Due to Lemma 1, the birth-death process has a finite number of states and thus a steady-state distribution. We argue that the process constitutes a Markov perfect equilibrium. As we explained in Remark 1, sellers are homogeneous once they join the market. Therefore, the dynamic game is stationary and Markovian with the number of sellers m in the marketplace as the state. To show that it is a Markov perfect equilibrium, it suffices to show that individual sellers do not have incentives to deviate from the equilibrium strategy. Since sellers can adjust the prices dynamically, we only need to show the strategy for state m, asking for a price p_m along with other sellers thanks to the price war, is optimal. For $m \geq 2$, asking for a price higher than p_m is clearly not optimal, because the arriving buyer will always choose other sellers offering a lower price p_m. On the other hand, undercutting p_m is not optimal because of the structural properties of (8) (See Lemma EC.2-(2)). For $m = 1$, the optimality of p_1 (f_0) is given by the HJB equation for f_1 in (8).

Next, we analyze the steady-state distribution of the Markov perfect equilibrium. To summarize the dynamics of the birth-death process, recall that the birth rate from state m to $m+1$ is $\lambda_s d_s(f_{m+1})$, $m = 0, \ldots, M-1$; the death rate from state m to $m-1$ is $\lambda_b d_b(p_m + c_b) = \lambda_b d_b(f_{m-1} + w)$, $m = 1, \ldots, M$. Therefore, the steady-state probability π_m for state m satisfies

$$\frac{\pi_m}{\pi_{m-1}} = \frac{\lambda_s d_s(f_m)}{\lambda_b d_b(f_{m-1} + w)}. \tag{10}$$

The next theorem provides the existence and uniqueness of the solution to (8) and hence the equilibrium.

THEOREM 1. *Suppose Assumption 1 and 2 hold. There exists a unique solution to* (8) *and thus a unique Markov perfect equilibrium of the dynamic game.*

The proof designs a *root-finding algorithm* that always outputs the solution f_m to (8). Finding the solution to nonlinear equations and proving the uniqueness is non-trivial, especially when the number of equations M in (8) is endogenous and part of the solution as well. Note that the equilibrium strategy of the sellers is constant given the state m. Thus, it is a measurable function of the state and the time, and the equilibrium is well-defined.

PROPOSITION 3. *In the equilibrium: (1) The expected utilities of sellers f_m as well as the transaction prices $p_m = f_{m-1} + c_s$, are decreasing and convex in m. (2) There exists $0 \le M^* < \infty$ such that when $m > M^*$, $\lambda_b d_b(p_m + c_b) \ge \lambda_s d_s(f_m)$; when $m \le M^*$, $\lambda_b d_b(p_m + c_b) < \lambda_s d_s(f_m)$. (3) The endogenous capacity M is decreasing in κ, λ_s, w, and increasing in λ_b. Moreover, there are sellers joining the marketplace if and only if $\max_p\{d_b(p+w)p\} \ge \kappa/\lambda_b$. (4) Given $w = c_s + c_b$, $V_{\mathcal{M}}(m)$ is increasing and concave in m and $\Delta V_{\mathcal{M}}(m) \le w$.*

The Sellers' Utility and Transaction Prices. The state m reflects the transient supply/demand imbalance. A larger m implies an excessive level of supply and each seller in the marketplace faces fiercer competition, which reduces the expected utility. The convexity of f_m implies that the adverse effect of competition is diminishing as m increases. This is because the sellers are selling homogeneous products. Intuitively, one can think of a single unit of demand being shared evenly among the sellers and the share $1/m$ is decreasing and convex in m.

The Mode of the Steady-State Distribution. Because f_m is decreasing, the effective arrival rate of a new seller, $\lambda_s d_s(f_{m+1})$, is decreasing in m; on the other hand, the arrival rate of a new buyer, $\lambda_b d_b(f_{m-1} + w)$, is increasing. It implies that the equilibrium dynamics exhibit "mean-reversion". That is, there is a state $M^* \triangleq \max\{m : \lambda_s d_s(f_m) \ge \lambda_b d_b(f_{m-1} + w)\}$ so that for $m > M^*$, the number of sellers m tends to decrease on average and for $m < M^*$ the number of sellers tends to increase. In other words, the platform is self-regulating the amount of supply because of the economic decisions made by the buyers and sellers. Combined with the property of π_m in (10), M^* is also the "mode" of the steady-state distribution.

The Endogenous Capacity. The endogenous capacity M is the maximal number of sellers the platform can potentially accommodate. By Proposition 3-(3), the endogenous capacity M is decreasing in the waiting cost κ, the market size of sellers λ_s, the total transaction fees w, and increasing in the market size of buyers λ_b. Consistent with the economic intuition, when the cost of selling the product decreases (low waiting cost or transaction fees) or the demand side becomes relatively large, the platform can accommodate more sellers. In extreme cases, for example, when the market size of buyers is considerably small, the endogenous capacity $M = 0$ and no sellers are willing to join the marketplace. The platform cannot make any revenue. Note that this is not the case in the dealership market: the dealer can always make profits no matter how small λ_b is.

The Platform's Revenue. This optimization problem is intractable for theoretical analysis, as $V_{\mathcal{M}}(m)$ depends on f_m, which cannot be solved explicitly and the dependence on the transaction fees is highly nonlinear. However, we can provide qualitative structural results about the platform's expected revenue, which are summarized in Proposition 3.

Although the platform does not keep its own inventory as the dealer does, the number of sellers m in the marketplace is connected to the inventory in the dealership model at a high level. In particular, the magnitudes of both quantities signal the excessive level of supply relative to demand and lead to lower transaction prices (see Proposition 1-(3) and Proposition 3-(1)). Interestingly, the value functions demonstrate similar structural properties (Proposition 1-(1) and Proposition 3-(4)). The platform makes more revenue as the number of sellers increases in the marketplace. This is because the firm's revenue only depends on the number of transactions regardless of the transaction price; more sellers endogenously lead to more transactions. Moreover, the marginal value is decreasing and bounded by w: an additional seller leads to at most one more transaction, and one transaction generates revenue w for the firm.

2.2.2. The Thick Market Consider a sequence of systems indexed by n. In the n-th system, $(\lambda_b^{(n)}, \lambda_s^{(n)}) = (n\lambda_b, n\lambda_s)$[7]. We characterize the limit of the dynamic game as $n \to \infty$, and derive the optimal strategy of the platform in the asymptotic regime.

As the market becomes thicker, transactions occur more frequently and the waiting time between the time points a seller joins the marketplace and her product gets sold becomes negligible. Therefore, sellers are more willing to join the platform and the endogenous capacity tends to infinity. Another important implication of a thick market is that the state transition rates of the dynamic game (birth-death process) become larger. As a result, the state of the game, m, resembles the state of a fluid system: the normalized inflow to the system is at the rate that a seller joins the marketplace $\lambda_s d_s(f_{m+1})$ and the outflow is at the rate that a buyer makes a purchase $\lambda_b d_b(f_{m-1} + w)$. Because the waiting cost diminishes, one has $f_m \approx f_{m+1} \approx f_{m-1}$. Therefore, define $\hat{f} \triangleq \left\{ f : \lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f} + w) \right\}$ to be the expected utility of a seller when the marketplace is in such a state that inflow equals the outflow. Intuitively, $\hat{f}$ is closely tied to the mode, M^*, of the distribution. Indeed, when there are M^* sellers in the marketplace, the inflow roughly matches the outflow and the expected utility f_{M^*} is very close to $\hat{f}$.

[7] Most quantities have the superscript (n) in this section. We omit it if there is no confusion.

Proposition 4. *In the thick-market regime: (1) For any finite* $k \in \mathbb{Z}$, $\lim_{n\to\infty} f^{(n)}_{M^*(n)+k} = \hat{f}$. *(2) For any* $\epsilon > 0$, *we have* $\lim_{n\to\infty} \sum_{m:|f^{(n)}_m - \hat{f}|>\epsilon} \pi_m = 0$.

Note that the standard techniques (*i.e.* fluid approximation) in revenue management cannot be used to prove the proposition, due to the price war among sellers. The first part shows that for a wide range of m, the expected utility of a seller, f_m, does not deviate much from $\hat{f}$. It is the key to characterizing the asymptotic steady-state distribution of the dynamic game. Because M^* is the mode of the steady-state distribution, the above result states that the expected utility of sellers is "flat" when the number of sellers is around the mode. Furthermore, the second part shows that the expected utility of a seller to join the marketplace is concentrated around $\hat{f}$ with high probability.[8] By Proposition 4, the expected utility of the sellers in the thick market can be approximated by $\hat{f}$ in the steady state. Hence the transaction prices are almost equal to $\hat{p} \triangleq \hat{f} + c_s$. The rate at which a transaction occurs is approximately $\lambda_b^{(n)} d_b(\hat{f} + c_s + c_b) = \lambda_b^{(n)} d_b(\hat{f} + w) = \lambda_s^{(n)} d_s(\hat{f})$, and thus the revenue rate is approximately $\lambda_s^{(n)} d_s(\hat{f}) w$.

When the market becomes thick, it is reasonable to focus on the long-run discounted revenue in the steady state, which is defined as $V_{\mathcal{M}} \triangleq \sum_{m=0}^{M} \pi_m V_{\mathcal{M}}(m)$. By (9), we have

$$
\begin{aligned}
rV_{\mathcal{M}} &= \sum_{m=0}^{M-1} \pi_m \lambda_s d_s(f_{m+1}) \Delta V_{\mathcal{M}}(m+1) + \sum_{m=1}^{M} \pi_m \lambda_b d_b(f_{m-1} + w)(w - \Delta V_{\mathcal{M}}(m)) \qquad (11) \\
&= w \sum_{m=1}^{M} \pi_{m-1} \lambda_s d_s(f_m) = w \sum_{m=1}^{M} \pi_m \lambda_b d_b(f_{m-1} + w),
\end{aligned}
$$

where we have used $\pi_{m-1}\lambda_s d_s(f_m) = \pi_m \lambda_b d_b(f_{m-1} + w)$ due to (10). Therefore, the optimal transaction fees of the firm are $\arg\max_{c_s,c_b} \frac{V^{(n)}_{\mathcal{M}}}{n} = \arg\max_{c_s,c_b} \{(c_s + c_b) \sum_{m=1}^{M^{(n)}} \pi_{m-1} \lambda_s d_s(f^{(n)}_m)\}$. This optimization is intractable for a finite n because $f^{(n)}_m$ and π_m do not have explicit forms and their dependence on c_s and c_b is highly nonlinear. However, given $w = c_s + c_b$, in the thick market, the firm's long-run average revenue converges, $\lim_{n\to\infty} \frac{V^{(n)}_{\mathcal{M}}}{n} = \frac{\lambda_s d_s(\hat{f}) w}{r}$ (see Proposition EC.2 in Section EC.3 in the online supplement). To maximize the revenue in the thick market, the firm sets c_s and c_b to maximize

$$
\max_{c_s,c_b} \quad \frac{1}{r} \lambda_s d_s(\hat{f})(c_s + c_b), \text{ subject to } \lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f} + c_s + c_b). \qquad (12)
$$

where $\hat{f}$ satisfies the "flow-balance" constraint. Surprisingly, the formulation has a fundamental connection to the optimization problem of the dealer in the thick market, which we discuss next.

[8] Note that Proposition 4 does not imply that $f_m \approx \hat{f}$ for all the states $m < M$. There are still infinitely many states m such that f_m is very different from $\hat{f}$ when $n \to \infty$.

3. Business Model Comparison

The dynamic models capture the differences in the two business models highlighted in Table 1. In the dealership market, the dealer has a monopolistic pricing power by setting bid and ask prices dynamically. In the marketplace, the firm only charges transaction fees that are set up upfront, and the transaction price is completely determined by the price war among sellers in the marketplace. The dealer manages his own inventory, and is thus exposed to the inventory risk. This is one of the reasons why the prices are set dynamically. In comparison, the platform is not exposed to inventory risk. The dealer makes a profit by earning the bid-ask spread, which straddles the marginal value of inventory. The platform earns revenues as long as buyers and sellers engage in transactions. For sellers, the benefit of selling to the dealer is the guaranteed immediate transactions, although it usually means that they are charged a premium by the dealer. Conversely, they are directly matched to the counterparty in the marketplace, after a (random) period of waiting. For buyers, they face a single dealer reselling the product in the dealership market, whereas the prices of the product are competitively set by the sellers in the marketplace.

REMARK 2. The differences in cash flows also play an important role when the market is thin. For any equilibrium prices f_m's induced by the transaction fee w in the marketplace model, we can construct the pricing strategy for a dealer with $p_s(x) = f_x$ and $p_b(x+1) = f_x + w$ for all $x \geq 0$, under which the behavior of sellers and buyers are exactly the same as that in the marketplace. However, due to the asynchronous bid-ask cash flows, the discounted profit of the dealer is lower. Consider m sellers (items) in the marketplace (dealer's inventory). As shown in Figure 2, in the marketplace, the seller arrives at t_1 and the buyer purchases it at $t_2 > t_1$. The cash flow for the platform is generated at t_2. In contrast, the dealer has to pay the seller at t_1 and then recoup it from the buyer at t_2. Because of discounting, the dealer's profit is lower than the platform's. Nevertheless, when the market is thick, $t_2 - t_1 \to 0$, the difference in the discounted cash flow is negligible.

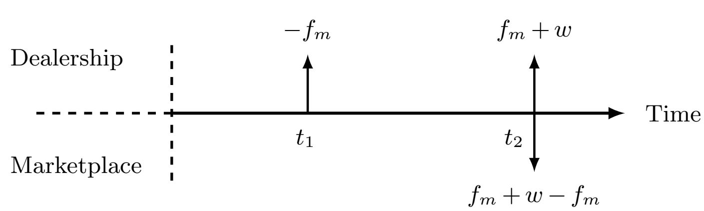


Figure 2 Illustration of cash flow for the dealership and marketplace.

3.1. A General Comparison between the Two Business Models

The profitability of the two business models can be quite different depending on the market environment. In particular, we can show that

Theorem 2. *Other parameters being fixed and $d_s(p) > 0$ for any $p > 0$, when the waiting cost of sellers κ is sufficiently large (above a certain threshold), or the market size of buyers λ_b is sufficiently small (below a certain threshold), the dealer earns more profits than the marketplace; when κ is sufficiently small (below a certain threshold), the marketplace earns more profits than the dealer.*

When sellers are impatient, they are unwilling to wait for long on the platform. As a result, not many sellers join and they are unable to offer a competitive price, which limits the transaction volume and thus the profitability of the marketplace relative to the dealership. In other words, when the market condition is not attractive to sellers, the marketplace cannot transfer the inventory risk to sellers and hence the dealer with the pricing power outperforms the marketplace. When the sellers are patient enough, a firm will transfer the inventory risk to the sellers and then earn more profit as a platform. Similarly, when not many buyers are populating the other side of the market, the marketplace looks less attractive to potential sellers, while the dealer can still wield its pricing power to partially balance the demand and supply. For other market conditions, we have found scenarios in which either business model can dominate the other in terms of revenues.

Despite the distinctions, the two models share several important features. We may regard the inventory held by the dealer and the number of sellers in the marketplace as similar indicators for transient excess supply in the two models. More inventory/sellers imply that the firm can meet short-term demand surges without worrying about the supply side. We have shown that the firm always benefits from higher inventory or more sellers but the marginal payoff is diminishing (see Proposition 1-(1) and Proposition 3-(4)). More inventory or sellers both lead to a lower transaction price (see Proposition 1-(3) and Proposition 3-(1)), though in the dealership market the transaction price is determined by the monopolistic dealer and in the marketplace it is an equilibrium outcome. Both quantities display mean reversion (see Proposition 1-(4) and Proposition 3-(2)), demonstrating the self-regulatory nature of the two models.

3.2. A Fundamental Connection in the Thick Market

For the dealer, (6) demonstrates the asymptotic optimality of constant bid and ask prices when the market is thick. For the platform, in the thick market the transaction fees are designed to maximize (12). The optimization problems exactly reflect the intuitions illustrated in Figure 1. In the thick market regime, both the inventory risk of the dealer and the waiting cost of the sellers in the marketplace are negligible. As a result, the demand and supply fluctuations and the intertemporal effects become second-order and the systems can be described by static optimization problems.

In Figure 1a, the dealer decides the bid and ask prices, i.e., p_s and p_b represented by E and F, respectively. The constraint $\lambda_s d_s(p_s) = \lambda_b d_b(p_b)$ implies that BC must be horizontal, i.e., the demand equals supply under the bid and ask prices. The objective $\lambda_s d_s(p_s)(p_b - p_s)/r$ is the area of the rectangle $BCEF$ divided by r. In Figure 1b, the platform charges the transaction fees c_s and c_b, causing the product to be cheaper and more expensive for sellers and buyers. It shifts the supply and demand curves to the new positions, represented by the dashed lines. The new equilibrium, in which the shifted demand equals the shifted supply, is represented by D. Interpret $\hat{f}$ as the x value of E or C. The constraint $\lambda_s d_s(\hat{f}) = \lambda_b d_b(\hat{f} + c_s + c_b)$ states that D is the new equilibrium after shift and BC is horizontal. The objective $\lambda_s d_s(\hat{f})(c_s + c_b)/r$ is the area of the rectangle $BCEF$ divided by r. In essence, in the thick market, the dealer and the platform are maximizing the same objective; their decisions, $\{p_s, p_b\}$ and $\{c_s, c_b\}$, also coincide $c_s + c_b = p_b - p_s$ at optimality and their revenues are equivalent after normalization. This observation leads to the next theorem.

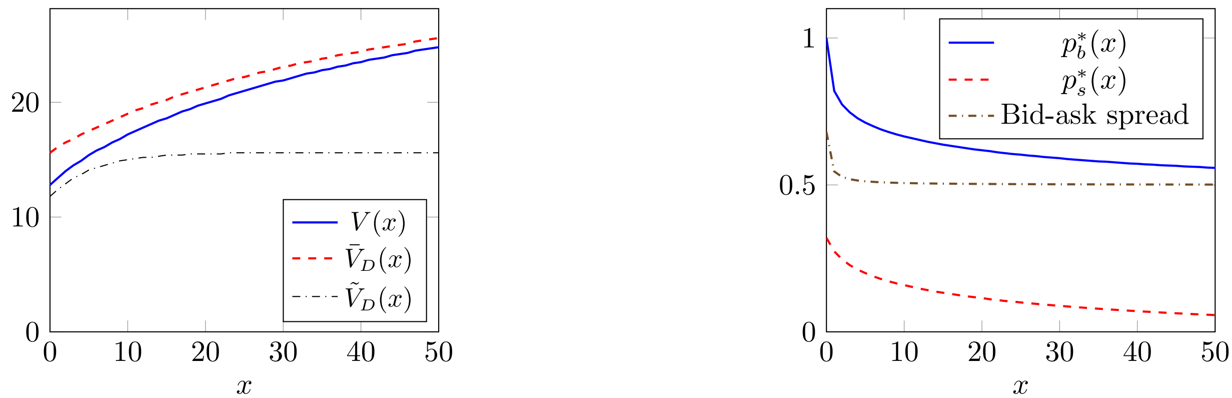
THEOREM 3. *In the limits of thick markets, two business models ((6) and (12)) share the following properties: (1) The revenues (normalized by the market size) are equivalent. (2) The optimal bid-ask spread of the dealer $p_b - p_s$ coincides with the commission fee of the platform $c_b + c_s$. (3) The optimal bid price set by the dealer coincides with the expected utility of sellers in the marketplace. (4) The optimal ask price set by the dealer coincides with the average payment made by buyers in the marketplace.*

Our model studies a homogeneous product where both sides consist of individual consumers. Therefore, in a thick market, the impact of each individual's decision on the entire market is negligible. This is one of the key reasons behind the equivalence. For example, the equivalence fails to hold in Abhishek et al. (2016), in which the supply side has only two manufacturers, and the dealer can soften the competition of the two manufacturers, compared to a marketplace.

4. Numerical Examples

In this section, we conduct a comprehensive numerical study based on linear demand and supply functions: $\lambda_b d_b(p) = \lambda_b(1-p)^+$ and $\lambda_s d_s(p) = \lambda_s \min\{p, 1\}$. Without loss of generality, we always set $\lambda_b = \lambda_s = \lambda$ which represents the market size. The main goal of this section is to explore some structural properties that cannot be investigated in a general model. Moreover, we test the robustness of our major insights for medium market sizes.

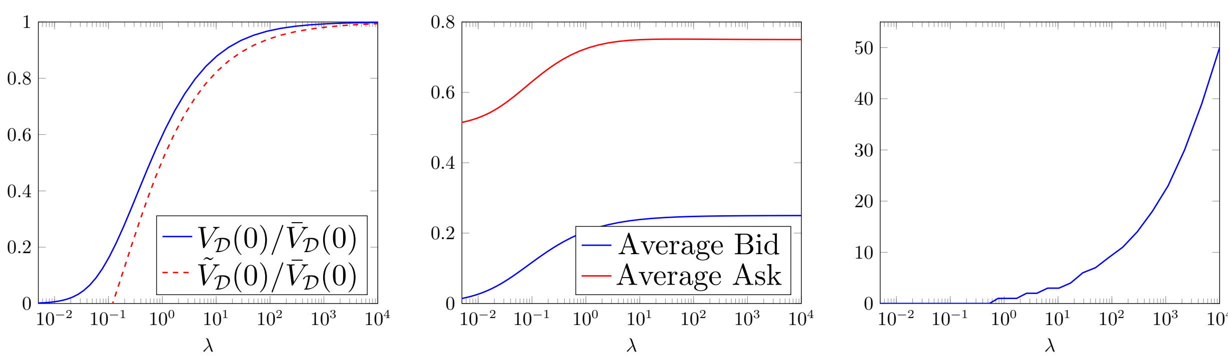
The Dealership Model. Figure 3 illustrates the structures of the optimal pricing policy. The underlying parameters are shown below. The discount factor per time unit for the dealer is $e^{-r} = 96.1\%$. On average, there are $\lambda = 5$ buyers and sellers per time unit arriving at the market. In a frictionless market, the equilibrium price is given by the intersection of the demand and supply curves $p_e = 0.5$. From panel (a), the value function of the fluid approximation is accurate for large x, while the fixed pricing policy performs well for small x. The bid-ask spread is decreasing and converging to 0.5 rapidly, although $p_b^*(x)$ and $p_s^*(x)$ are converging to their limits relatively slowly. The preferred inventory position (defined in Proposition 1) is given by the intersection of the arrival rates of buyers and sellers, $x_p = 3$.



(a) The value functions for the optimal policy, the fluid approximation, and the fixed-pricing policy

(b) The optimal pricing policy

Figure 3 **The optimal solution for the dealership model with linear demand and supply** $\lambda_b d_b(p) = 5(1-p)^+$**,** $\lambda_s d_s(p) = 5\min\{p, 1\}$ **and** $r = 0.04$**.**



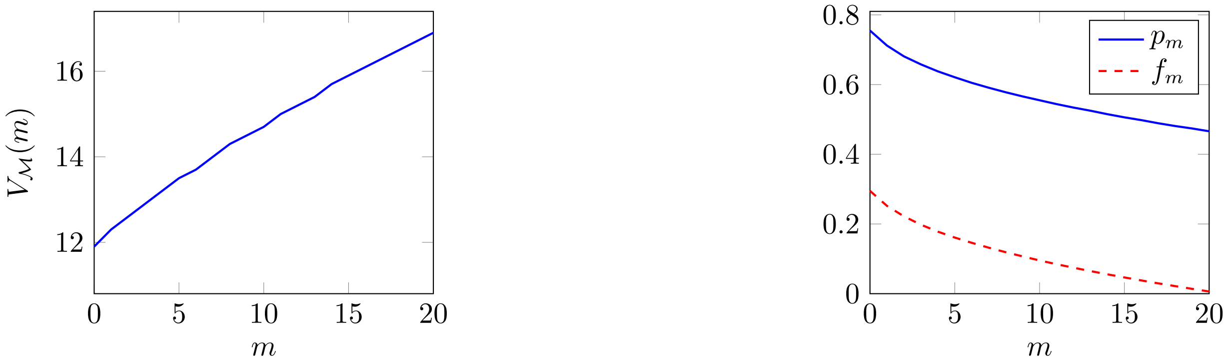
(a) The value functions of the optimal and fixed-pricing policies relative to the fluid approximation

(b) The average bid and ask price in the steady state

(c) The preferred inventory level: x_p

Figure 4 **Some performance measures for the dealership model with** $\lambda_b d_b(p) = \lambda(1-p)^+$**,** $\lambda_s d_s(p) = \lambda \min\{p, 1\}$ **and** $r = 0.04$**.**

Next, we investigate the performance of the fixed-pricing policy and other quantities of the optimal pricing policy for different market sizes λ. The parameters in Figure 4 are the same as those in Figure 3. Figure 4a demonstrates $V(0)/\bar{V}_D(0)$ and $\tilde{V}_D(0)/\bar{V}_D(0)$. When the market is sufficiently thick ($\lambda > 1000$), both the optimal pricing policy and fixed-pricing policy lead to a similar revenue relative to that of the fluid approximation. In Figure 4b,we plot the average bid and ask prices in the steady state.[9] Both prices increase in the market size and finally converge. Figure 4c shows that the preferred inventory position of the dealer increases as the market becomes thicker. A thick market leads to stronger stochastic fluctuations, and requires the dealer to have a larger buffer.



(a) The platform's revenue under the optimal policy

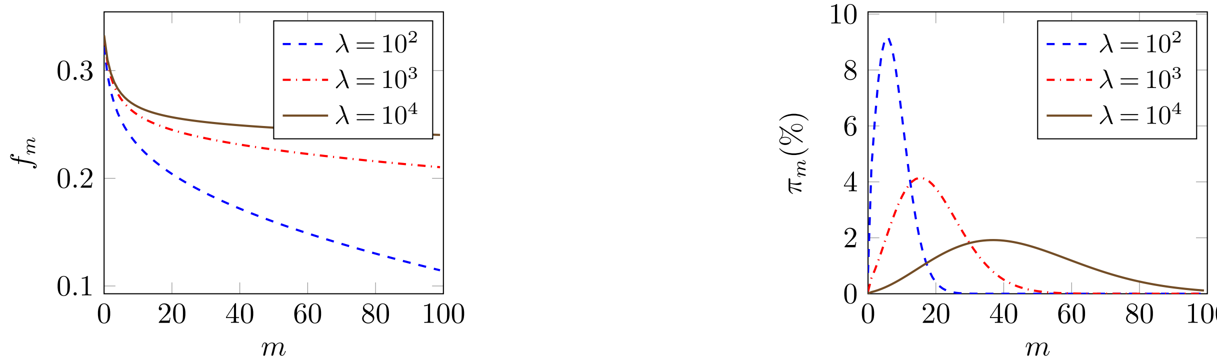
(b) The seller's expected utility and the payment by buyers under the optimal policy

Figure 5 **The optimal solution for the marketplace with linear demand** $\lambda_b d_b(p) = 5(1-p)^+$**,** $\lambda_s d_s(p) = 5\min\{p, 1\}$**,** $r = 0.04$ **and** $\kappa = 0.02$**.**

The Marketplace. We use the same set of parameters r and λ as in Figure 3. The sellers have a waiting cost $\kappa = 0.02$ per time unit before their items are sold. Figure 5 illustrates the structure of the optimal pricing policy. The optimal fee is approximately 0.46. When the charged fee is low, the platform makes little profit per transaction; when the charged fee is high, the expected number of transactions decreases. The optimal transaction fees charged by the platform is close to the thick-market policy 0.5, suggested by (12). From Figure 5a, we can see under the optimal policy, the platform's revenue $V_{\mathcal{M}}(m)$ is increasing and concave in the number of sellers in the marketplace m. The monotonicity and convexity of the seller's expected utility f_m and the payment made by the buyers in m can be observed in Figure 5b. The mode of the steady-state distribution of sellers is given by the intersection of the arrival rates of buyers and sellers, i.e., $M^* = 1$.

Figure 6 illustrates the convergence (see Proposition 4) when the market sizes grow. From the figures, it is clear that the steady-state distribution does not become more concentrated. It is the combination of "flatter" f_m and π_m that leads to our technical characterization of the convergence in Proposition 4.

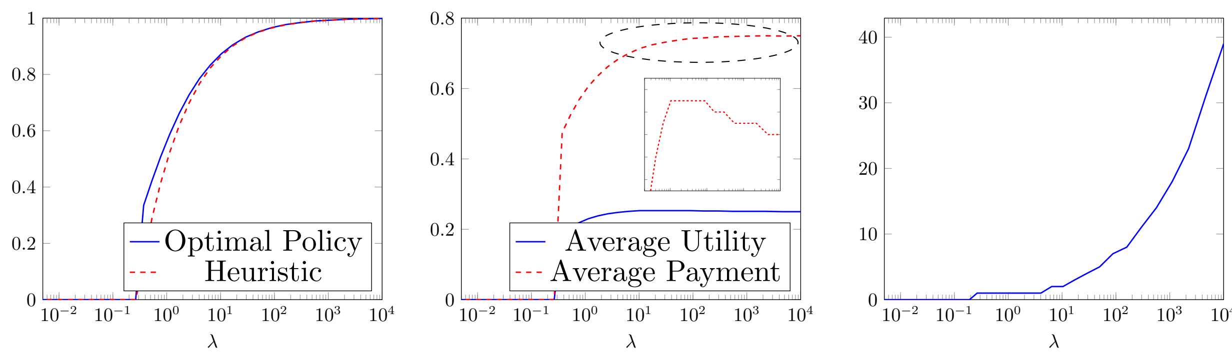
[9] Since the bid and ask prices are set based on the current inventory, we take the average when the inventory reaches the steady state under the optimal policy.



(a) The seller's expected utility f_m

(b) The steady-state distribution $\pi_m(\%)$

Figure 6 **Illustration of the convergence in the marketplace in the thick market with** $\lambda_b d_b(p) = \lambda(1-p)^+$**,** $\lambda_s d_s(p) = \lambda \min\{p, 1\}$**,** $r = 0.04$ **and** $\kappa = 0.02$**. Note for** $\lambda = 10^4$ **the mode** $M^* = 39$ **and** $\hat{f} \approx 0.25$**.**



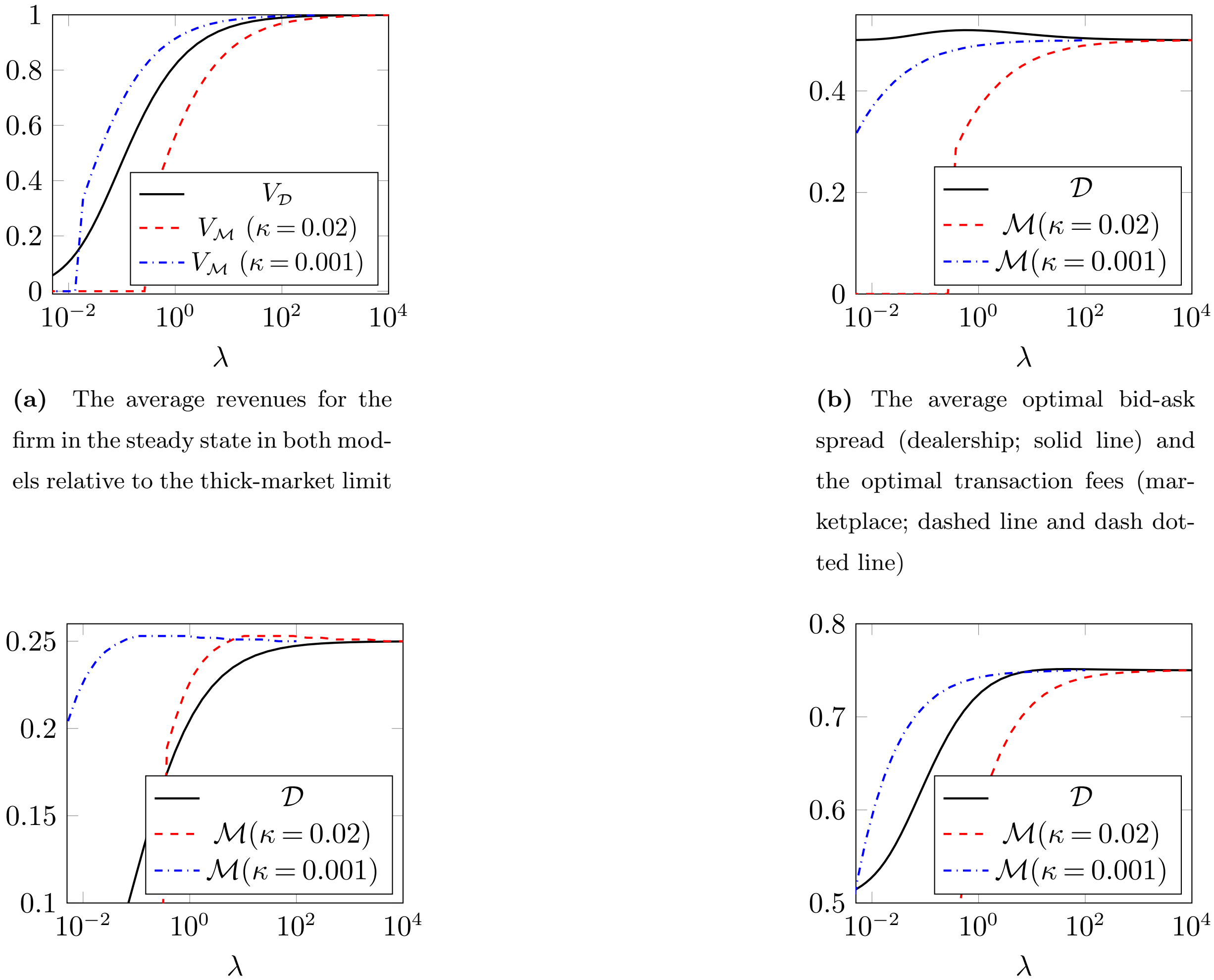
(a) The long-run average revenue of the optimal policy and the thick-market heuristic relative to the thick market limit

(b) The average utility of a seller $\sum_m \pi_m f_m$ and the average payment made by buyers $\sum_m \pi_m (p_m + c_b)$.

(c) The long-run mean of the number of sellers

Figure 7 **Various quantities in the marketplace for different market sizes.** $\lambda_b d_b(p) = \lambda(1-p)^+$**,** $\lambda_s d_s(p) = \lambda \min\{p, 1\}$**,** $r = 0.04$ **and** $\kappa = 0.02$**.**

We proceed to investigate the performance of the thick-market heuristic and other quantities for different market sizes λ. Figure 7a demonstrates the platform's long-run average revenue of the optimal policy and the thick-market policy, which performs well even for medium market sizes. The pattern is similar to Figure 4a and validates Theorem 3 that both business models are equally profitable in a thick market. As shown in Figure 7b, both the seller's average utility and the average payment of a buyer converge as the market size grows. However, the expansion of the market does not uniformly benefit the sellers and their utility may decrease as a result. This is caused by the heightened competition among sellers: there are more potential sellers when the market size



(a) The average revenues for the firm in the steady state in both models relative to the thick-market limit

(b) The average optimal bid-ask spread (dealership; solid line) and the optimal transaction fees (marketplace; dashed line and dash dotted line)

(c) The average bid price (dealership) and utility of a seller (marketplace)

(d) The average ask price (dealership) and payment made by buyers (marketplace)

Figure 8 **Comparison of two models with** $\lambda_b d_b(p) = \lambda(1-p)^+$**,** $\lambda_s d_s(p) = \lambda \min\{p, 1\}$ **and** $r = 0.04$**.**

increases and they are more likely to join thanks to the higher demand rate. In other words, the growth of the market may intensify the competition and thus hurt existing sellers on the platform.

Figure 7c shows that the average number of sellers also increases, as they see more potential profits from the increasing number of buyers. This is consistent with Figure 4c where the dealer also stocks more inventory as λ increases.

Finally, we compare the two business models under the optimal policies. The underlying parameters of Figure 8 and Table 3 are the same as those in Figure 7. As the theory predicts, when the market is thick, the firm's revenues in both models converge to the same limit. For the set of parameters ($\kappa = 0.02$), the firm is always better off acting as a dealer due to the waiting cost κ, which is a free parameter the dealership model does not have. If we set $\kappa = 0.001$ instead, then the revenue of the marketplace dominates the dealer's profit except for a very small λ.

In Figure 8b, we compare the bid-ask spread of the dealer and the optimal fee of a platform. Both quantities converge to the same value when the market size grows. The optimal transaction fees are always increasing in the market size, however, the average bid-ask spread is not necessarily monotone. In Figure 8c and Figure 8d, we compare a few measures related to the surplus of consumers: In the dealership model, we plot the average bid and ask prices experienced by buyers and sellers; in the marketplace, we plot the average utility of a seller (f_m averaged over the steady state) and the average payment made by buyers (the cost of the product plus the transaction fee paid to the platform). We can see that in general, the prices are rising when the market size grows or the sellers become more patient.

$(\lambda_b+\lambda_s)/2$ \ λ_b/λ_s	0.001	0.01	0.1	1	10	100	1000
0.001	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
0.003	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.01	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.03	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.3	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
3	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
10	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
30	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
100	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$
300	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$
1000	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$

Table 3 The more profitable business model, depending on the market size $\lambda_b + \lambda_s$ and the composition of buyers and sellers λ_b/λ_s. The notation $\mathcal{D}/\mathcal{M}$ indicates that both models are equally profitable, with a revenue difference less than 2%.

In Table 3, we show which business model is more profitable with $\kappa = 0.001$, for different market sizes and compositions of buyers and sellers. Consistent with Theorem 2 and Theorem 3, when the market size is sufficiently large, these two models are equally profitable. When the market size of buyers is relatively small, the dealership model is more profitable. When the market size of sellers is relatively small, the marketplace is more profitable.

Consumer Surplus. Next, we numerically explore the consumer surplus.

The Dealership. As Ω_b and Ω_s are the random valuations of buyers and sellers, once the dealer purchases one product from a seller at price p_s, the expected surplus of the seller is $\mathbb{E}[p_s - \Omega_s \mid \Omega_s \leq p_s]$. Similarly, the expected surplus of the buyer is $\mathbb{E}[\Omega_b - p_b \mid \Omega_b \geq p_b]$ for price p_b. Therefore, given the stationary distribution $\pi(\cdot)$ and the optimal prices $p_b(\cdot)$ and $p_s(\cdot)$, we can compute the long-run average consumer surplus as

$$\mathcal{S}_{\mathcal{D}} = \sum_{i=0}^{\infty} \pi(i)\Big(\lambda_b d_b(p_b(i))\mathbb{E}[\Omega_b - p_b(i) \mid \Omega_b \geq p_b(i)] + \lambda_s d_s(p_s(i))\mathbb{E}[p_s(i) - \Omega_s \mid \Omega_s \leq p_s(i)]\Big).$$

The Marketplace. Once a seller joins the market with m sellers, her expected surplus is $\mathbb{E}[f_{m+1} - \Omega_s \mid \Omega_s \leq f_{m+1}]$. Once a buyer makes a purchase with m sellers, her expected surplus is $\mathbb{E}[\Omega_b - f_{m-1} - w \mid \Omega_b \geq f_{m-1} + w]$. Given the stationary distribution $\pi(\cdot)$, the transaction fee w and the expected utility f_m, we can compute the long-run average consumer surplus as

$$\begin{aligned}\mathcal{S}_{\mathcal{M}} = \sum_{i=1}^{M} \pi(i)\Big(&\lambda_b d_b(f_{i-1} + w)\mathbb{E}[\Omega_b - f_{i-1} - w \mid \Omega_b \geq f_{i-1} + w] \\ &+ \lambda_s d_s(f_{i+1})\mathbb{E}[f_{i+1} - \Omega_s \mid \Omega_s \leq f_{i+1}]\Big) + \pi(0)\lambda_s d_s(f_1)\mathbb{E}[f_1 - \Omega_s \mid \Omega_s \leq f_1].\end{aligned}$$

Using the same parameters as Table 3, Table 4 shows which business model benefits consumers more. Similar to our discussion of Theorem 2, when the market size of buyers is relatively small, the market is not attractive to sellers and the resulting low transaction volume limits consumer surplus in the marketplace. When the market size is large or the market mainly consists of buyers, the marketplace not only generates higher profits for the platform but also induces higher consumer surplus. Compared with the dealership, the marketplace extracts less consumer surplus (bid-ask spread/transaction fee) per transaction (see Figure 8b) but benefits from the larger transaction volume, leading to higher consumer surplus. Specifically, due to the inventory risk, the dealer conservatively purchases buffer inventory from sellers and exploits its pricing power to earn a large bid-ask spread, resulting in a relatively low transaction volume. In contrast, given a high demand rate and the small waiting cost, the marketplace can attract sizable sellers waiting on the platform, resulting in large transaction volume.

5. Extensions

5.1. Exponential Discounting and Percentage Fees

In this section, we show how to extend our base model in Section 2.2 to incorporate exponential discounting and percentage fees. In particular, the seller's utility joining the marketplace of m sellers in (7) is replaced by $f_m = \mathbb{E}[e^{-r_s\tau}\tilde{p}]$, where r_s is the discount factor for sellers. We also assume that the fee is proportional to the transaction price and fully paid by the buyer. That is, when the ask price is p on the marketplace, the buyer pays a total of $(1+\alpha)p$, among which αp is paid to the platform. To derive the HJB equation, note that within the next dt period,

$$\begin{aligned}f_m = (1 - r_s dt)\Big(&\lambda_b d_b((1+\alpha)f_{m-1})dt f_{m-1} + \lambda_s d_s(f_{m+1})dt f_{m+1} \\ &+ \big(1 - \lambda_b d_b((1+\alpha)f_{m-1})dt - \lambda_s d_s(f_{m+1})dt\big) f_m\Big),\end{aligned} \tag{13}$$

λ_b/λ_s \ $(\lambda_b+\lambda_s)/2$	0.001	0.01	0.1	1	10	100	1000
0.001	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
0.003	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.01	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.03	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
0.3	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
1	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
3	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
10	$\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
30	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
100	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
300	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$
1000	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$

Table 4 The business model with higher consumer surplus for different market sizes and supply-demand imbalances.

for $m \geq 2$. That is, the first term is the expected utility when a transaction happens and the number of sellers decreases by one, the second term refers to the event that a new seller joins, and the third term is the event that the number of sellers doesn't change within the period. Similar to (8), it allows us to derive the system of equations that characterize f_m:

$$f_1 = \max_{f_0} \left\{ \frac{\lambda_b d_b((1+\alpha)f_0)f_0 + \lambda_s d_s(f_2)f_2}{r_s + \lambda_b d_b((1+\alpha)f_0) + \lambda_s d_s(f_2)} \right\}$$
$$f_m = \frac{\lambda_b d_b((1+\alpha)f_{m-1})f_{m-1} + \lambda_s d_s(f_{m+1})f_{m+1}}{r_s + \lambda_b d_b((1+\alpha)f_{m-1}) + \lambda_s d_s(f_{m+1})}, \; m \geq 2.$$

Similar to the base model, given the values of f_m, we can formulate the equations for the platform's discounted revenue $V_{\mathcal{M}}(m)$ when there are m sellers in the marketplace:

$$rV_{\mathcal{M}}(0) = \lambda_s d_s(f_1)\Delta V_{\mathcal{M}}(1)$$
$$rV_{\mathcal{M}}(m) = \lambda_s d_s(f_{m+1})\Delta V_{\mathcal{M}}(m+1) + \lambda_b d_b((1+\alpha)f_{m-1})(\alpha f_{m-1} - \Delta V_{\mathcal{M}}(m)), \; \forall m \geq 1.$$

To ensure the finiteness of the marketplace, we impose Assumption 2. Moreover, we assume that there exists a minimum positive price for the seller to accept.

Assumption 3. *There exists $\underline{p} > 0$ such that $d_s(\underline{p}) = 0$.*

Under Assumptions 1, 2 and 3, we can show that Lemma 1 still holds. In particular, the endogenous capacity of the marketplace M satisfies $M \leq 2\lambda_b \ln(\bar{p}/\underline{p})/r_s$. In fact, we can show that the seller's utility of joining a marketplace with m sellers is upper bounded by

$$f_m = \mathbb{E}[e^{-r_s\tau}\tilde{p}] \leq e^{-r_s\mathbb{E}[\tau]}\bar{p} \leq e^{-\frac{r_s(m+1)}{2\lambda_b}}\bar{p},$$

where the second inequality follows from Jensen's inequality and the last inequality follows from $\mathbb{E}[\tau] \geq \frac{m+1}{2\lambda_b}$ (see the proof for Lemma 1 in the online supplement). As a result, when $m > 2\lambda_b \ln(\bar{p}/\underline{p})/r_s$, the seller's utility of joining is less than $\underline{p}$ and thus no new sellers are joining.

The Thick Market. After solving f_m and $V_{\mathcal{M}}(m)$, we can calculate the steady-state discounted revenue for the platform:

$$\begin{aligned} rV_{\mathcal{M}} &= \sum_{m=0}^{M-1} \pi_m \lambda_s d_s(f_{m+1}) \Delta V_{\mathcal{M}}(m+1) + \sum_{m=1}^{M} \pi_m \lambda_b d_b((1+\alpha) f_{m-1})(\alpha f_{m-1} - \Delta V_{\mathcal{M}}(m)) \\ &= \sum_{m=1}^{M} \pi_m \lambda_b d_b((1+\alpha) f_{m-1})(\alpha f_{m-1}), \end{aligned}$$

where the steady-state probability satisfies $\pi_{m-1}\lambda_s d_s(f_m) = \pi_m \lambda_b d_b((1+\alpha) f_{m-1} + w)$. In the thick market, it is easy to show that $V_{\mathcal{M}} \to \frac{1}{r}\lambda_b d_b\big((1+\alpha)\hat{f}\big)(\alpha \hat{f})$, and we have formulated a similar revenue-maximizing problem to (12):

$$\max_{\alpha, w \geq 0} \frac{1}{r} \lambda_s d_s(\hat{f})(\alpha \hat{f}), \quad \text{subject to } \lambda_s d_s(\hat{f}) = \lambda_b d_b((1+\alpha)\hat{f}), \tag{14}$$

It is clear that the optimal revenue of the platform in the thick market remains the same under exponential discounting and percentage fees. The result suggests that as the market becomes thicker, there is less need for the marketplace to extensively examine the selection of the fee structure (fixed/proportional) and sellers' waiting behavior (linear/exponential waiting cost). In this case, the pattern in Table 3 can still be observed in the numerical experiments and the insights are the same as the base model. Moreover, similar to Theorem 2, when $r_s > 2\lambda_b \ln(\bar{p}/\underline{p})$ or $\lambda_b < \frac{r_s}{2\ln(\bar{p}/\underline{p})}$, we can prove that the dealer earns more profits than the marketplace.

5.2. Vertically Differentiated Products

Consider two types of vertically differentiated products.[10] The arrival rates of sellers holding these two products are $\lambda_{s,1}$ and $\lambda_{s,2}$, and their private WTS for product $i \in \{1,2\}$ are denoted by $\Omega_{s,i}$. Therefore, the expected supply is given by $d_{s,i}(p) \triangleq \mathsf{P}(\Omega_{s,i} \leq p)$. For buyers, their valuations for the products are $(\theta_1\Omega_b, \theta_2\Omega_b)$, where Ω_b is random and (θ_1, θ_2) are known. One can interpret (θ_1, θ_2) as the quality of the two products and Ω_b as the random sensitivity to quality. Thus, given price (p_1, p_2), the demand for product one is $d_{b,1}(p_1, p_2) \triangleq \mathsf{P}(\theta_1\Omega_b - p_1 \geq \theta_2\Omega_b - p_2;\ \theta_1\Omega_b - p_1 \geq 0)$. The demand for product two can be defined similarly. The arrival rate of the buyers is λ_b.

The Dealership Market. In the dealership market, when the current inventory levels are $x_1, x_2 \in \mathbb{N}$, the dealer uses the bid and ask prices $p_{b,i}(t)$ and $p_{s,i}(t)$ to maximize the expected revenue over an infinite horizon discounted at rate r. To formulate the HJB equation, we let

[10] The formulation can be further extended to the n-product case. To overcome the curse of dimensionality, we may use approximate dynamic programming to speed up the computation.

$\Delta_1 V_{\mathcal{D}}(x_1,x_2) = V_{\mathcal{D}}(x_1,x_2) - V_{\mathcal{D}}(x_1-1,x_2)$ and $\Delta_2 V_{\mathcal{D}}(x_1,x_2) = V_{\mathcal{D}}(x_1,x_2) - V_{\mathcal{D}}(x_1,x_2-1)$ be the marginal values of product 1 and 2, respectively. Then, the value functions satisfy the following set of equations:

$$\begin{aligned}
rV_{\mathcal{D}}(x_1,x_2) =& \lambda_b \max_{p_{b,1},p_{b,2}\ge 0} \left\{ d_{b,1}(p_{b,1},p_{b,2})(p_{b,1} - \Delta_1 V_{\mathcal{D}}(x_1,x_2)) + d_{b,2}(p_{b,1},p_{b,2})(p_{b,2} - \Delta_2 V_{\mathcal{D}}(x_1,x_2)) \right\} \\
&+ \lambda_{s,1} \max_{p_{s,1}\ge 0} d_{s,1}(p_{s,1})(\Delta_1 V_{\mathcal{D}}(x_1+1,x_2) - p_{s,1}) \\
&+ \lambda_{s,2} \max_{p_{s,2}\ge 0} d_{s,2}(p_{s,2})(\Delta_2 V_{\mathcal{D}}(x_1,x_2+1) - p_{s,2}), \quad x_1,x_2 \ge 1 \\
rV_{\mathcal{D}}(0,x_2) =& \lambda_b \max_{p_{b,2}\ge 0} \left\{ d_{b,2}(\infty,p_{b,2})(p_{b,2} - \Delta_2 V_{\mathcal{D}}(x_1,x_2)) \right\} + \lambda_{s,1} \max_{p_{s,1}\ge 0} d_{s,1}(p_{s,1})(\Delta_1 V_{\mathcal{D}}(x_1+1,x_2) - p_{s,1}) \\
&+ \lambda_{s,2} \max_{p_{s,2}\ge 0} d_{s,2}(p_{s,2})(\Delta_2 V_{\mathcal{D}}(x_1,x_2+1) - p_{s,2}), \qquad x_2 \ge 1 \\
rV_{\mathcal{D}}(x_1,0) =& \lambda_b \max_{p_{b,1}\ge 0} \left\{ d_{b,1}(p_{b,1},\infty)(p_{b,1} - \Delta_1 V_{\mathcal{D}}(x_1,x_2)) \right\} + \lambda_{s,1} \max_{p_{s,1}\ge 0} d_{s,1}(p_{s,1})(\Delta_1 V_{\mathcal{D}}(x_1+1,x_2) - p_{s,1}) \\
&+ \lambda_{s,2} \max_{p_{s,2}\ge 0} d_{s,2}(p_{s,2})(\Delta_2 V_{\mathcal{D}}(x_1,x_2+1) - p_{s,2}), \qquad x_1 \ge 1 \\
rV_{\mathcal{D}}(0,0) =& \lambda_{s,1} \max_{p_{s,1}\ge 0} d_{s,1}(p_{s,1})(\Delta_1 V_{\mathcal{D}}(x_1+1,x_2) - p_{s,1}) + \lambda_{s,2} \max_{p_{s,2}\ge 0} d_{s,2}(p_{s,2})(\Delta_2 V_{\mathcal{D}}(x_1,x_2+1) - p_{s,2}).
\end{aligned}$$

Similar to the base model, we consider the optimization problem in the fluid approximation.

$$\begin{aligned}
\bar{V}(0,0) = & \max_{p_{s,1},p_{s,2},p_{b,1},p_{b,2}} \frac{1}{r}\Big(\lambda_{s,1} d_{s,1}(p_{s,1})(p_{b,1} - p_{s,1}) + \lambda_{s,2} d_{s,2}(p_{s,2})(p_{b,2} - p_{s,2}) \Big) \\
& s.t. \quad \lambda_{s,1} d_{s,1}(p_{s,1}) = \lambda_b d_{b,1}(p_{b,1},p_{b,2}), \qquad \lambda_{s,2} d_{s,2}(p_{s,2}) = \lambda_b d_{b,2}(p_{b,1},p_{b,2}).
\end{aligned}$$

To interpret it, note that the constraints require the dealer to set the bid and ask prices for both products to balance their supply and demand. The objective is the discounted profit from selling both products.

The Marketplace. For the marketplace, we let $f^i_{x_1,x_2}$ be the expected utility of a seller selling product i (type-i seller) when there are x_i type-i sellers. Because the sellers can adjust prices continuously, by the same argument as the base model, we must have $p^1_{x_1,x_2} = f^1_{x_1-1,x_2}$ and $p^2_{x_1,x_2} = f^2_{x_1,x_2-1}$, where $p^1_{x_1,x_2}$ and $p^2_{x_1,x_2}$ are the ask prices for the two products. We also let $p^2_{x_1,0} = \infty$ and $p^1_{0,x_2} = \infty$ by convention. The firm charges w_i for the transaction of product i. Similar to (8), we have a set of recursive equations characterizing the values of $f^i_{x_1,x_2}$ shown in Section EC.10 of the online supplement. With $f^i_{x_1,x_2}$, one can derive the value functions for the platform $V_{\mathcal{M}}(x_1,x_2)$ as well as the discounted revenue in the steady state $V_{\mathcal{M}}$.

We conduct a numerical experiment to compare the profitability of the two business models. We consider different market sizes (measured by $\lambda_b + \lambda_s$) and supply-demand imbalances (measured by λ_b/λ_s). We set $\Omega_b \backsim U[0,1]$, $\Omega_{s,i} \backsim U[0,1]$, $\lambda_{s,1} = \lambda_{s,2} = \lambda_s$ and $(\theta_1,\theta_2) = (1, 0.5)$. Table 5 shows the more profitable business model under different conditions. The pattern is similar to Table 3: The marketplace is more profitable when the market size is larger.

$(\lambda_b+\lambda_s)/2$ \ λ_b/λ_s	0.001	0.01	0.1	1	10	100	1000
0.001	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
0.003	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
0.01	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$
0.03	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$
0.1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$
0.3	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
3	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
10	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
30	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
100	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$
300	$\mathcal{D}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{M}$
1000	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$	$\mathcal{D}/\mathcal{M}$

Table 5 The more profitable business model for different market sizes and supply-demand imbalances.

5.3. Thin Market: A Special Case

In this section, we study a special thin market to complement our analysis of the thick market. In the thin market, because of the scarce supply and demand, the dealer holds at most one item because the waiting cost for a future buyer in the form of discounting outweighs the benefit of holding more inventory. Similarly, at most one seller will participate in the marketplace. For ease of analysis, we assume that the demand functions of buyers and sellers are $d_b(p) = (b_1 - p)^+$ and $d_s(p) = \mathbf{1}_{p \geq b_2}$, respectively.[11]

The Dealership Market. Following the notation in Section 2.1, we derive the HJB equation as follows:

$$
\begin{aligned}
rV_{\mathcal{D}}(0) &= \lambda_s \max_{p_s \geq 0} \left\{ \mathbf{1}_{p_s \geq b_2} (\Delta V_{\mathcal{D}}(1) - p_s) \right\} \\
rV_{\mathcal{D}}(1) &= \lambda_s \max_{p_s \geq 0} \left\{ \mathbf{1}_{p_s \geq b_2} (\Delta V_{\mathcal{D}}(2) - p_s) \right\} + \lambda_b \max_{p_b \geq 0} \left\{ (b_1 - p_b)^+ (p_b - \Delta V_{\mathcal{D}}(1)) \right\} \\
rV_{\mathcal{D}}(x) &= \lambda_s \max_{p_s \geq 0} \left\{ \mathbf{1}_{p_s \geq b_2} (\Delta V_{\mathcal{D}}(x+1) - p_s) \right\} + \lambda_b \max_{p_b \geq 0} \left\{ (b_1 - p_b)^+ (p_b - \Delta V_{\mathcal{D}}(x)) \right\}.
\end{aligned}
$$

where recall that $\Delta V(x)$ represents the marginal value of one unit of inventory when the current inventory is x. As a result, when $\Delta V_{\mathcal{D}}(1) \geq b_2 \geq \Delta V_{\mathcal{D}}(2)$, it is optimal for the dealer to set an ask price to buy from the seller when he holds no inventory because the marginal value of the first unit is higher than the buyers' valuation. On the other hand, the marginal value of the second unit is lower than the valuation of the buyers, and it is never optimal to buy the second unit. We can show that this condition is met when λ_b falls in a range.

[11] The indicator function can also be approximated by a continuous function satisfying Assumption 1.

Lemma 2. *When $\lambda_b \in [0, \frac{4b_2 r}{(b_1-b_2)^2})$, the dealer does not hold inventory and the long-run discounted revenue in the steady state is $V_{\mathcal{D}}^* = 0$. When $\lambda_b \in [\frac{4b_2 r}{(b_1-b_2)^2}, \frac{8b_2 r}{(b_1-b_2)^2}]$, the dealer holds at most one unit of inventory and the long-run discounted revenue in the steady state is*

$$V_{\mathcal{D}}^* = \frac{\lambda_s}{\lambda_b r\left(\varsigma - r\right)} b_2 \lambda_b \left(-\varsigma + 2\lambda_s + r\right) + 2\left(\lambda_s + r\right)\left(\varsigma - \lambda_s - r\right) + b_1 \lambda_b \left(\varsigma - 2\lambda_s - 2r\right), \tag{15}$$

where $\varsigma = \sqrt{b_1\lambda_b(\lambda_s + r) - b_2\lambda_b\lambda_s + (\lambda_s + r)^2}$.

The Marketplace. We observe a similar phenomenon: when the arrival rate of the buyers is small, only one seller is willing to join the marketplace. That is, the endogenous capacity is $M = 1$. More precisely, we can show that

Lemma 3. *When $\lambda_b \in [0, \frac{4\kappa}{(b_1-b_2)^2})$, there is no seller in the marketplace and the long-run discounted revenue for the platform in the steady state is $V_{\mathcal{M}}^* = 0$. When $\lambda_b \in [\frac{4\kappa}{(b_1-b_2)^2}, \frac{25\kappa}{4(b_1-b_2)^2}]$, there is at most one seller in the marketplace and the long-run discounted revenue for the platform in the steady state is*

$$V_{\mathcal{M}}^* = \frac{\sqrt{\kappa\lambda_b}\lambda_s\left(b_1 - b_2 - 2\sqrt{\frac{\kappa}{\lambda_b}}\right)}{\sqrt{\kappa\lambda_b} r + \lambda_s r}, \tag{16}$$

and the optimal fee is $w^* = b_1 - b_2 - 2\sqrt{\frac{\kappa}{\lambda_b}}$.

Lemmas 2 and 3 allow us to compare the two business models.

Lemma 4. *Suppose the conditions of Lemmas 2 and 3 are satisfied. Fixing the total market size $\lambda_b + \lambda_s$, we have: (1) The optimal bid-ask spread $p_b^*(1) - p_s^*(1)$ and the optimal transaction fee increase in the demand-supply imbalance λ_b/λ_s. (2) When λ_b/λ_s is sufficiently small, the preferred business model is the dealership; when $\kappa < \frac{r(2r + b_1\lambda - 2\sqrt{r(r + b_1\lambda)})}{4\lambda}$ and λ_b/λ_s is sufficiently large, the preferred business model is the marketplace.*

From Lemma 4, we are able to derive similar results to Theorem 2 and validate the pattern in Table 1 theoretically, which can only be numerically demonstrated without the thin-market conditions. In particular, when the market size is fixed, the imbalance of the demand side over the supply side favors the marketplace model.

6. Conclusion

We build dynamic models for the two most common business models in two-sided markets: dealership and marketplace. For the marketplace, we use Markovian dynamic games to capture the waiting cost of sellers without resorting to queueing models. The competitive nature of the sellers allows the framework to yield a tractable transient analysis of the equilibrium. Although having distinct features and facing different trade-offs, the two business models exhibit a fundamental connection, illustrated by Figure 1. In particular, when the two sides of the market get thicker, the optimal revenue earned from the two models converge and the optimal pricing/commission policies can be simply translated to each other. Our study provides new economic insights when comparing the two business models. We also provide a few case studies and economic insights in Section EC.5 in the online supplement.

References

Abhishek V, Jerath K, Zhang ZJ (2016) Agency selling or reselling? Channel structures in electronic retailing. *Management Science* 62(8):2259–2280.

Amihud Y, Mendelson H (1980) Dealership market: Market-making with inventory. *Journal of Financial Economics* 8(1):31–53.

Bai J, So KC, Tang CS, Chen X, Wang H (2019) Coordinating supply and demand on an on-demand service platform with impatient customers. *Manufacturing & Service Operations Management* 21(3):479–711.

Benjaafar S, Ding JY, Kong G, Taylor T (2022) Labor welfare in on-demand service platforms. *Manufacturing & Service Operations Management* 24(1):110–124.

Besbes O, Castro F, Lobel I (2021) Surge pricing and its spatial supply response. *Management Science* 67(3):1350–1367.

Bimpikis K, Candogan O, Saban D (2019) Spatial pricing in ride-sharing networks. *Operations Research* 67(3):744–769.

Birge J, Candogan O, Chen H, Saban D (2021) Optimal commissions and subscriptions in networked markets. *Manufacturing & Service Operations Management* 23(3):569–588.

Chen N, Gallego G (2019) Welfare analysis of dynamic pricing. *Management Science* 65(1):139–151.

Chen X, Simchi-Levi D (2004a) Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research* 52(6):887–896.

Chen X, Simchi-Levi D (2004b) Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research* 29(3):698–723.

Chen Y, Hu M (2020) Pricing and matching with forward-looking buyers and sellers. *Manufacturing & Service Operations Management* 22(4):717–734.

Cohen MC, Zhang R (2022) Competition and coopetition for two-sided platforms. *Production and Operations Management* 31(5):1997–2014.

Cusumano M, Gawer A, Yoffie D (2019) *The Business of Platforms: Strategy in the Age of Digital Competition, Innovation, and Power* (Harper Collins), ISBN 9780062896322.

Duffie D, Garleanu N, Pedersen LH (2005) Over-the-counter markets. *Econometrica* 73(6):1815–1847.

Gallego G, Topaloglu H (2019) *Revenue Management and Pricing Analytics* (Springer-Verlag New York).

Gallego G, van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science* 40(8):999–1020.

Gautier P, Hu B, Watanabe M (2023) Marketmaking middlemen. *The RAND Journal of Economics* 54(1):83–103.

Ghuloum M, Song B, Aydin G, Souza GC (2022) Joint dynamic pricing with acquisition and selling opportunities. *European Journal of Operational Research* 297(1):252–267.

Hagiu A, Wright J (2024) Marketplace leakage. *Management Science* 70(3):1529–1553.

He EJ, Goh J (2022) Profit or growth? Dynamic order allocation in a hybrid workforce. *Management Science* 68(8):5891–5906.

Hu M, Zhou Y (2017) Price, wage and fixed commission in on-demand matching. Technical report, University of Toronto.

Li F, Murry C, Tian C, Zhou Y (2019) The price theory and empirics of inventory management. *SSRN Electronic Journal* .

Lian Z, Van Ryzin G (2021) Optimal growth in two-sided markets. *Management Science* 67(11):6862–6879.

Light B, Johari R, Weintraub G (2024) Quality selection in two-sided markets: A constrained price discrimination approach. *Operations Research* Ahead of Print.

Manheim (2015) 2015 used car market report. Technical report, Manheim.

Mayhew S (2002) Competition, market structure, and bid-ask spreads in stock option markets. *The Journal of Finance* 57(2):931–958.

Mortensen DT, Wright R (2002) Competitive pricing and efficiency in search equilibrium. *International Economic Review* 43(1):1–20.

Njoroge P, Ozdaglar A, Stier-Moses NE, Weintraub GY (2014) Investment in two-sided markets and the net neutrality debate. *Review of Network Economics* 12(4):355–402.

Parker GG, Van Alstyne MW, Choudary SP (2016) *Platform Revolution: How Networked Markets Are Transforming the Economy–And How to Make Them Work for You* (W. W. Norton & Company, New York), 1st edition, ISBN 0393249131.

Rochet J, Tirole J (2003) Platform competition in two-sided markets. *Journal of European Economic Association* 1(4):990–1029.

Rochet J, Tirole J (2006) Two-sided markets: A progress report. *The RAND Journal of Economics* 37(3):645–667.

Rust J, Hall G (2003) Middlemen versus market makers: A theory of competitive exchange. *Journal of Political Economy* 111(2):353–403.

Satterthwaite M, Shneyerov A (2007) Dynamic matching, two-sided incomplete information, and participation costs: Existence and convergence to perfect competition. *Econometrica* 75(1):155–200.

Taylor TA (2018) On-demand service platforms. *Manufacturing & Service Operations Management* 20(4):704–720.

Acknowledgments

The authors gratefully thank the Department Editor (Marcelo Olivares), the Associate Editor, and two referees for their diligent reading and thoughtful comments that significantly improved the paper.

Online Supplement for "Dealership or Marketplace"

EC.1. Remarks for Assumption 1 and Proposition 1

Assumption 1. The first assumption is common in the literature of dynamic pricing, e.g., Gallego and van Ryzin (1994). The second assumption guarantees the existence of a unique equilibrium price $p_e > 0$ such that $\lambda_b d_b(p_e) = \lambda_s d_s(p_e)$; in a competitive equilibrium without dealers, it is given by the intersection of demand and supply curves. The last technical assumption is that the suprema are attained in the definitions of $\mathcal{R}_b$ and $\mathcal{R}_s$. A sufficient condition is that $d_b(p)$ and $d_s(p)$ are continuous in $p \geq 0$ and $d_b(p) = o(1/p)$. Most classic demand (supply) functions satisfy this condition, e.g., linear demand and exponential demand. For any $z > 0$, define $P_b(z) \triangleq \arg\max_{p\geq 0} \mathcal{R}_b(p,z)$ and $P_s(z) \triangleq \arg\max_{p\geq 0} \mathcal{R}_s(p,z)$. Elementary comparative statics reveal that $P_b(z)$ and $P_s(z)$ are nonnegative and increasing in $z \geq 0$. Note that this standard assumption also guarantees the sufficiency and necessity of the HJB equation, similar to Gallego and van Ryzin (1994), Ghuloum et al. (2022).

Proposition 1. In our model, the bid-ask spread arises from the concavity of $V_{\mathcal{D}}(x)$ as it implies that $p_b^*(x) \geq \Delta V_{\mathcal{D}}(x) \geq \Delta V_{\mathcal{D}}(x+1) \geq p_s^*(x)$, i.e., the marginal value of inventory must be straddled by the spread. Note that the bid-ask spread does not necessarily straddle the equilibrium price p_e, at which demand and supply cross, i.e., $\lambda_b d_b(p_e) = \lambda_s d_s(p_e)$. The monotonicity of $p_s^*(x)$ in the dealer's inventory level is intuitive: As x increases, the marginal value of inventory decreases and the dealer is unwilling to buy more, thus the bid price decreases. Combining with $\lim_{x\to\infty} p_s^*(x) = 0$, it implies that if the lowest valuation of sellers is above 0, there are no sellers willing to sell to the dealer when the dealer's inventory is higher than a threshold. The monotonicity of $p_b^*(x)$ in the dealer's inventory level follows from Gallego and van Ryzin (1994): The ask price increases in the "scarcity" of inventory, as $\Delta V_{\mathcal{D}}(x)$ decreases in x. However, the bid-ask spread is *not* monotone in x in general. With unlimited inventory, the dealer does not need to replenish the inventory from the sellers and simply optimizes the revenue rate from the buyers. When the interest rate increases (or the market sizes decrease), the dealer lowers the bid and ask prices to avoid the high inventory risk, or maintain the turnover rate. The mean reversion of the inventory is consistent with the findings in Amihud and Mendelson (1980), Li et al. (2019). This phenomenon occurs due to the inventory risk: The dealer is unwilling to keep a very low level of inventory, as potential buyers might be turned away if $x = 0$ due to stochastic fluctuation; on the other hand, he is unwilling to stock too much, as the inventory risk is high because of the discounting. As the market size increases (λ_b and λ_s increase), the dealer needs more inventory as a "buffer" to counter the increasing stochastic fluctuation. This explains the monotonicity of x_p. It implies that for a thicker market, the dealer is willing to keep more inventory to hedge the inventory risk.

EC.2. Proofs for Section 2.1

Proof of Proposition 1: Part one is just the same as part one of Proposition EC.3 in Section EC.7. Recall that $p_b^*(x) = \arg\max_{p\geq 0}\{d_b(p)(p - \Delta V_{\mathcal{D}}(x))\}$, and $p_s^*(x) = \arg\max_{p\geq 0}\{d_s(p)(\Delta V_{\mathcal{D}}(x+1) - p)\}$. For part two, it is straightforward as $p_s^*(x) \leq \Delta V_{\mathcal{D}}(x+1) \leq \Delta V_{\mathcal{D}}(x) \leq p_b^*(x)$.

For part three, as $\lim_{x\to\infty} \Delta V_{\mathcal{D}}(x) = 0$, $\lim_{x\to\infty} p_s^*(x) = 0$ and $\lim_{x\to\infty} p_b^*(x) = \arg\max_p\{d_b(p)p\}$. From Proposition EC.3, $\Delta V_{\mathcal{D}}(x)$ is decreasing in x and $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$. Then by Lemma EC.3 in Section EC.7, we have the monotonicity of $p_b^*(x)$ and $p_s^*(x)$. The dependence on the market size when λ_b and λ_s change in proportion, is just the opposite to that of r.

The properties about the preferred inventory level follow directly from the monotonicity of $p_s^*(x)$ and $p_b^*(x)$. □

Then, we show the properties of the value function of the fluid approximation.

PROPOSITION EC.1. *The value function $\bar{V}_{\mathcal{D}}(x)$ satisfies the following properties:*

1. *$\bar{V}_{\mathcal{D}}(x)$ is increasing and concave in x.*
2. *$\bar{V}_{\mathcal{D}}(x)$ is an upper bound of $V_{\mathcal{D}}(x)$ and $\lim_{x\to\infty} \bar{V}_{\mathcal{D}}(x) = \lim_{x\to\infty} V_{\mathcal{D}}(x) = \lambda_b \mathcal{R}_b(0)/r$.*

Proof of Proposition EC.1: By the coupling argument, it is easy to prove that $\bar{V}_{\mathcal{D}}(x)$ is strictly increasing in x. For concavity, note that we must have $\lambda_b d_b(\bar{p}_b^*(x)) \geq \lambda_s d_s(\bar{p}_s^*(x))$ for all $x \geq 0$, i.e., the inventory is always being depleted. This is because in the fluid approximation, there is no randomness and it is never optimal to stock more inventory. Because $\bar{V}_{\mathcal{D}}(x)$ is strictly increasing in x, we must have that for any $x > 0$,

$$0 < \frac{dr\bar{V}_{\mathcal{D}}(x)}{dx} = \frac{d[\lambda_b \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x))]}{dx} = [\lambda_s d_s(\bar{p}_s^*(x)) - \lambda_b d_b(\bar{p}_b^*(x))] V''_{\mathcal{D}}(x),$$

where the second equality follows from the envelope theorem. Therefore, $\bar{V}''_{\mathcal{D}}(x) < 0$ and $\bar{V}_{\mathcal{D}}(x)$ is increasing concave.

For part two, it is easy to show that $\lim_{x\to\infty} \bar{V}_{\mathcal{D}}(x) = \lambda_b \mathcal{R}_b(0)/r$.

Next, we show that $\bar{V}(x)$ is an upper bound for $V(x)$. Because $\bar{V}(x)$ is concave, $\Delta\bar{V}_{\mathcal{D}}(x) > \bar{V}'_{\mathcal{D}}(x) > \Delta\bar{V}_{\mathcal{D}}(x+1)$. By the property of $\mathcal{R}_b$ and $\mathcal{R}_s$ in Lemma EC.3 in Section EC.7, we have

$$r\bar{V}_{\mathcal{D}}(x) = \lambda_b \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(x)) \geq \lambda_b \mathcal{R}_b(\Delta\bar{V}_{\mathcal{D}}(x)) + \lambda_s \mathcal{R}_s(\Delta\bar{V}_{\mathcal{D}}(x+1)),$$

for $x \in \mathbb{Z}^+$. Moreover, by the continuous differentiability of $\bar{V}_{\mathcal{D}}$, we have

$$r\bar{V}_{\mathcal{D}}(0) = \lambda_b \mathcal{R}_b(\bar{V}'_{\mathcal{D}}(0)) + \lambda_s \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(0)) \geq \lambda_s \mathcal{R}_s(\bar{V}'_{\mathcal{D}}(0)) \geq \lambda_s \mathcal{R}_s(\Delta\bar{V}_{\mathcal{D}}(1)).$$

The HJB equation in (1) can be formulated as linear programming,

$$\begin{aligned} V_{\mathcal{D}}(y) = \min_{F(\cdot)} \quad & F(y) \\ \text{subject to} \quad & rF(0) \geq \lambda_s \mathcal{R}_s(\Delta F(1)), \\ & rF(x) \geq \lambda_b \mathcal{R}_b(\Delta F(x)) + \lambda_s \mathcal{R}_s(\Delta F(x+1)), \text{ for any } x \in \mathbb{Z}^+. \end{aligned}$$

Clearly, $\bar{V}_{\mathcal{D}}(x)$ is a feasible solution to the problem above, then $V_{\mathcal{D}}(x) \le \bar{V}_{\mathcal{D}}(x)$. $\square$

Proof of Proposition 2: When the dealer adopts the fixed pricing policies in (6), the dynamics of the inventory follow that of a state-independent M/M/1 queue. Denote $\tilde{V}_{\mathcal{D}}(x)$ as the expected discounted profit for this policy with x units of initial inventory. By the Markovian structure, we have

$$\begin{aligned}(r+\lambda_b d_b(\hat{p}_b)+\lambda_s d_s(\hat{p}_s))\tilde{V}_{\mathcal{D}}(x) &= \lambda_b d_b(\hat{p}_b)\tilde{V}_{\mathcal{D}}(x-1)+\lambda_s d_s(\hat{p}_s)\tilde{V}_{\mathcal{D}}(x+1)\\ &\quad +\hat{p}_b\lambda_b d_b(\hat{p}_b)-\hat{p}_s\lambda_s d_s(\hat{p}_s) \quad x>0,\\ (r+\lambda_s d_s(\hat{p}_s))\tilde{V}_{\mathcal{D}}(0) &= \lambda_s d_s(\hat{p}_s)\tilde{V}_{\mathcal{D}}(1)-\hat{p}_s\lambda_s d_s(\hat{p}_s).\end{aligned} \tag{EC.1}$$

From the first equation above, $\tilde{V}_{\mathcal{D}}(x)$ must be of the form $\tilde{V}_{\mathcal{D}}(x)=C_0 s_0^x + C_1 s_1^x + \frac{1}{r}(\hat{p}_b\lambda_b d_b(\hat{p}_b) - \hat{p}_s\lambda_s d_s(\hat{p}_s))$, where $0<s_0<1<s_1$ and s_0 and s_1 are the two solutions to $\lambda_s d_s(\hat{p}_s)x^2-(r+\lambda_b d_b(\hat{p}_b)+\lambda_s d_s(\hat{p}_s))x+\lambda_b d_b(\hat{p}_b)=0$, with

$$s_0=\frac{r+\lambda_b d_b(\hat{p}_b)+\lambda_s d_s(\hat{p}_s)-\sqrt{(r+\lambda_b d_b(\hat{p}_b)+\lambda_s d_s(\hat{p}_s))^2-4\lambda_s\lambda_b d_s(\hat{p}_s)d_b(\hat{p}_b)}}{2\lambda_s d_s(\hat{p}_s)}\in(0,1).$$

Since $\tilde{V}_{\mathcal{D}}(x)$ is bounded above by the same argument as in Proposition EC.3 in Section EC.7, we must have $C_1=0$. By the boundary condition, the last equation in (EC.1),

$$C_0=-\frac{\hat{p}_b\lambda_b d_b(\hat{p}_b)}{r+\lambda_s d_s(\hat{p}_s)(1-s_0)}<0.$$

Now scale demand and supply simultaneously by n, i.e., $(\lambda_b^n,\lambda_s^n)=n(\lambda_b,\lambda_s)$ and denote $V_D^{(n)}(x)$,$\bar{V}_D^{(n)}(x)$,$\tilde{V}_D^{(n)}(x)$ as the corresponding value function in the original problem, fluid approximation and the fixed pricing policy, respectively. Denote $d_n=n\lambda_b d_b(\hat{p}_b)=n\lambda_s d_s(\hat{p}_s)$. Then

$$|\tilde{V}_{\mathcal{D}}^{(n)}(x)-\bar{V}_{\mathcal{D}}^{(n)}(0)|=|C_0|s_0^x\le|C_0|=\frac{\hat{p}_b d_n}{r+\left(\sqrt{r^2+4rd_n}-r\right)/2}=O(\sqrt{n}).$$

As $\bar{V}_{\mathcal{D}}^{(n)}(x)=\Omega(n)$, $\lim_{n\to\infty}\frac{\tilde{V}_{\mathcal{D}}^{(n)}(x)}{\bar{V}_{\mathcal{D}}^{(n)}(0)}=1$ and $\tilde{V}_{\mathcal{D}}^{(n)}(x)\le V_{\mathcal{D}}^{(n)}(x)\le\bar{V}_{\mathcal{D}}^{(n)}(x)$, we have the result. $\square$

EC.3. Proofs for Section 2.2

Proof of Lemma 1: Given Assumptions 1 and 2, we first provide a lower bound for the expected waiting time when there are m sellers, $\mathbb{E}[\tau_m]$. Consider a case when no new sellers will come, buyers arrive with the largest effective rate λ_b, and all sellers have the same chance to sell their products. In this case, each seller's expected waiting time is

$$\sum_{i=1}^m \mathbb{P}(\text{sell to } i\text{-th buyer})\mathbb{E}[\text{Arrival Time of the } i\text{-th buyer}]\ge\sum_{i=1}^m\frac{1}{m}\frac{i}{\lambda_b}=\frac{m+1}{2\lambda_b}, \tag{EC.2}$$

which is apparently a lower bound for $\mathbb{E}[\tau_m]$.

Then, the capacity M can be upper bounded by a constant $2\bar{p}\lambda_b/\kappa$ because when $m \geq 2\bar{p}\lambda_b/\kappa - 1$, we have

$$f_m = \mathbb{E}[\tilde{p}_m - c_s - \kappa\tau_m] \leq \bar{p} - \kappa\mathbb{E}[\tau_m] \leq \bar{p} - \kappa\frac{m+1}{2\lambda_b} \leq 0,$$

where the first inequality holds because $\tilde{p} \leq \bar{p}$ and $c_s \geq 0$. The second inequality holds due to (EC.2). □

EC.3.1. Proof of Theorem 1

We first introduce some notations, and a few auxiliary results used in the proof. Define f^* as the unique solution to $\lambda_b\tilde{\mathcal{R}}_b(f^*) = \lambda_b \max_{p\geq 0}\{d_b(p+w)(p-f^*)\} = \kappa$.[12] Define f^{**} as the smallest positive solution to $\lambda_b d_b(f^{**}+w)f^{**} = \kappa$ if there exists such a solution or $+\infty$ otherwise. Note that when $f^* > 0$, $f^{**} < +\infty$ exists and $\frac{d\lambda_b d_b(p+w)p}{dp}|_{p=f^{**}} > 0$.[13] We also define the following function

$$R(x; y, z) \triangleq \lambda_b d_b(x+w)(x-y) - \kappa - \lambda_s d_s(z)(y-z). \tag{EC.3}$$

Using this function, the recursive system (8) can be rewritten as

$$\begin{aligned} &\sup_x R(x; f_1, f_2) = 0, \\ &R(f_{m-1}; f_m, f_{m+1}) = 0, m \geq 2. \end{aligned} \tag{EC.4}$$

Now we are ready to prove Theorem 1. Note that from the definitions of f^* and f^{**}, we only need to establish the proof for three separate cases: $f^* \leq 0$, $0 < f^* \leq f^{**}$ and $f^* > f^{**} > 0$. The proof is organized as follows: In Section EC.3.1.1, we prove the existence of solutions to (8) when $f^* \leq 0$ or $0 < f^* \leq f^{**}$. In Section EC.3.1.2, we prove the existence of solutions to (8) when $f^* > f^{**} > 0$. In Section EC.3.1.3, we demonstrate the properties that any solution to (8) will satisfy. Using these properties, we then prove that the solutions that have been found in Section EC.3.1.1 and Section EC.3.1.2 are the unique solutions to (8). Thus there exists a unique Markov perfect equilibrium of the dynamic game.

EC.3.1.1. Existence: $f^* \leq 0$ or $0 < f^* \leq f^{**}$. When $f^* \leq 0$ or $0 < f^* \leq f^{**}$, then we can show that

$$f_1 = f^*, \quad f_m = f_{m-1} - \frac{\kappa}{\lambda_b d_b(f_{m-1}+w)} \quad m \geq 2 \tag{EC.5}$$

[12] To show that f^* always exists, note that $\lim_{z\to+\infty}\tilde{\mathcal{R}}_b(z) = 0$ and $\lim_{z\to-\infty}\tilde{\mathcal{R}}_b(z) = +\infty$, and that $\tilde{\mathcal{R}}_b(z)$ is continuous. To show the uniqueness, note that by the similar proof in Lemma EC.3 in Section EC.7, $\tilde{\mathcal{R}}_b(\cdot)$ is monotone. Moreover, $\tilde{\mathcal{R}}_b'(z)|_{z=f^*} = -\lambda_b d_b(P_b(f^*)+w) < 0$, so $\tilde{\mathcal{R}}_b(z)$ is strictly decreasing at $z = f^*$. This implies that f^* is unique.

[13] If $f^* > 0$, then $\max_p\{\lambda_b d_b(p+w)p\} > \max_p\{\lambda_b d_b(p+w)(p-f^*)\} = \kappa$. Therefore, f^{**} exists. Moreover, by Assumption 1, $\lambda_b d_b(p+w)p$ is unimodal in $p \in [0,\infty)$. Because f^{**} is the smaller solution, we have $\frac{d\lambda_b d_b(p+w)p}{dp}|_{p=f^{**}} > 0$.

is a solution to (8). In fact, if $f^* \leq 0$, then $f_2 \leq f_1 = f^* \leq 0$. Therefore, $M = 0$ and (8) degenerates to a single equation $\kappa = \max_{f_0}\{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}$, which holds when $f_1 = f^*$ by the definition of f^*.

If $0 < f^* \leq f^{**}$, then from the definition of f^{**} (see Footnote 13), $f^* \leq f^{**} < p^* \triangleq \arg\max_{p\geq 0}\{\lambda_b d_b(p+w)p\}$. By Assumption 1, $\lambda_b d_b(p+w)p$ is increasing in $[0, p^*)$ and thus $\lambda_b d_b(f^* + w)f^* \leq \lambda_b d_b(f^{**} + w)f^{**} = \kappa$ (by definition of f^{**}). Therefore, $f_2 = f^* - \frac{\kappa}{\lambda_b d_b(f^*+w)} \leq 0$, so $M = 1$ and (8) again degenerates to a single equation $\kappa = \max_{f_0}\{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}$, which holds for $f_1 = f^*$.

EC.3.1.2. Existence: $f^* > f^{**} > 0$. To show the existence of solutions to (8) when $f^* > f^{**} > 0$, we design Algorithm 1 (see Section EC.6 in the additional material). The algorithm is based on the following intuition: Assign a value to f_M, then recursively compute f_{M-1}, f_{M-2}, ..., f_1 using the Mth, $(M-1)$th, ..., second equation of (8). Then check whether the first equation of (8) holds. Since M is not known in advance, in the algorithm, we use X_n to denote f_{M+1-n}. When backwards calculating X_{n+1}, we use $R(X_{n+1}; X_n, X_{n-1}) = 0$ due to (EC.4). The additional constraint in the computation $\frac{dR(x;X_n,X_{n-1})}{dx}|_{x=X_{n+1}} > 0$ will be specified in Lemma EC.2 as a necessary condition for any solution to (8).

The algorithm takes an input a, which is a candidate solution for f_M. If the output is $R_N = 0$, $N < \infty$, $X_0 \leq 0$ and $X_i \geq 0$ for $i > 0$, then $[X_N, \cdots, X_1, X_0]$ must satisfy

$$\sup_x R(x; X_N, X_{N-1}) = 0,$$
$$R(f_{n+1}; f_n, f_{n-1}) = 0, \quad 1 \leq n < N.$$

It solves (EC.4), or equivalently (8) , by substituting $M = N$, $f_1 = X_N$, $\cdots$, $f_M = X_1$, $f_{M+1} = X_0$.

Denote $N(a)$ as the output of N when the input is a. Similarly, we introduce $X_n(a)$ and $R_n(a)$ for fixed n. First, we point out some facts for the proposed algorithm, based on Assumption 1 and Lemma 1 and the following conditions: $f^* \geq f^{**} > 0$ and $a \in [0, f^{**}]$. These facts are then used to prove the existence of solutions to (8) when $f^* > f^{**} > 0$ based on Algorithm 1.

Fact 1: $X_0(a) \leq 0$, $X_0(a) < X_1(a)$ and $\frac{d\lambda_b d_b(x+w)(x-X_0(a))}{dx}|_{x=X_1(a)} > 0$.

This fact states that if $f_M = a \in [0, f^{**}]$, then $f_{M+1} = X_0(a) \leq 0$. So the choice of a is proper.

Proof of Fact 1: By their definitions in the algorithm, $X_1(a) = a \geq 0$ and $X_0(a) < X_1(a)$. Let $p^* \triangleq \arg\max_{p\geq 0}\{\lambda_b d_b(p + w)p\}$. By the definition (Footnote 13), $f^{**} < p^*$. By Assumption 1, $\lambda_b d_b(p + w)p$ is increasing in $[0, p^*)$. Then as $a \leq f^{**} < p^*$, $\lambda_b d_b(a + w)a \leq \lambda_b d_b(f^{**} + w)f^{**} = \kappa$ and thus $X_0(a) = a - \frac{\kappa}{\lambda_b d_b(a+w)} \leq 0$, where the equality holds only when $a = f^{**}$. When $a = f^{**}$, $\frac{d\lambda_b d_b(x+w)(x-X_0(a))}{dx}|_{x=X_1(a)} = \frac{d\lambda_b d_b(x+w)x}{dx}|_{x=f^{**}} > 0$, where the inequality is due to the definition of f^{**}. When $a < f^{**}$, $X_0(a) < 0$ and thus

$$\lambda_b d_b(X_1(a) + w)(X_1(a) - X_0(a)) = \kappa = \lambda_b d_b(f^{**} + w)f^{**} < \lambda_b d_b(f^{**} + w)(f^{**} - X_0(a)),$$

where the first equality is due to the definition of $X_0(a)$ in the algorithm and the second equality is due to the definition of f^{**}. Due to Assumption 1, $\lambda_b d_b(x+w)(x-X_0(a))$ is unimodal in x and as $X_1(a)=a<f^{**}$, the inequality above leads to $\frac{d\lambda_b d_b(x+w)(x-X_0(a))}{dx}|_{x=X_1(a)}>0$. $\square$

Fact 2: If $R_n(a)>0$, $X_{n+1}(a)$ exists and is unique. Moreover, $X_{n+1}(a)\geq X_n(a)+\frac{\kappa}{\lambda_b d_b(w)}>X_n(a)\geq 0$.

This fact states that for any input $a\in[0,f^{**}]$, there is a unique output by the algorithm.

Proof of Fact 2: As $X_1(a)=a\geq 0\geq X_0(a)$, we have $R(0;X_1(a),X_0(a))<0$. By Assumption 1, $R(x;X_1(a),X_0(a))$ is unimodal in $x\in[0,\infty)$. Then if $R_1(a)=\sup_{x\geq 0}R(x;X_1(a),X_0(a))>0$, there exists a unique $X_2(a)>0$ satisfying $R(X_2(a);X_1(a),X_0(a))=0$ and $\frac{dR(x;X_1(a),X_0(a))}{dx}|_{x=X_2(a)}>0$. As $R(X_2(a);X_1(a),X_0(a))=\lambda_b d_b(X_2(a)+w)(X_2(a)-X_1(a))-\kappa-\lambda_s d_s(X_0(a))(X_1(a)-X_0(a))=0$ and $X_1(a)>X_0(a)$, we have $\lambda_b d_b(X_2(a)+w)(X_2(a)-X_1(a))-\kappa\geq 0$. That is, $X_2(a)-X_1(a)\geq \frac{\kappa}{\lambda_b d_b(X_2(a)+w)}>\frac{\kappa}{\lambda_b d_b(w)}>0$. By induction, if $R_n(a)>0$, $X_{n+1}(a)$ exists and is unique. Besides, $X_{n+1}(a)\geq X_n(a)+\frac{\kappa}{\lambda_b d_b(w)}>X_n(a)\geq 0$. $\square$

Fact 3: $N(f^{**})=N(0)-1>0$, $R_{N(0)}(0)=R_{N(0)-1}(f^{**})\leq 0$ and $R_{N(0)-1}(0)>0$.

This fact checks the property of two extremes of the input $a\in[0,f^{**}]$.

Proof of Fact 3: By the step in the algorithm, $R_1(0)=\sup_x R(x;0,-\frac{\kappa}{\lambda_b d_b(w)})=\max_p\{\lambda_b d_b(p+w)p\}-\kappa>0$, where the inequality is due to $f^*>0$. From the algorithm, it will lead to $N(0)>1$. Clearly, $X_0(f^{**})=0$, $X_1(f^{**})=f^{**}$, $X_1(0)=0$ and $X_2(0)=f^{**}$. Therefore, there is always a lag of one when using the input $a=0$ and $a=f^{**}$. One can check that $X_n(0)=X_{n-1}(f^{**})$ and thus $N(f^{**})=N(0)-1>0$. From the condition that the algorithm stops, we have $R_{N(0)}(0)=R_{N(f^{**})}(f^{**})=R_{N(0)-1}(f^{**})\leq 0$ and $R_{N(0)-1}(0)>0$. $\square$

Fact 4: $N(a)<+\infty$.

Proof of Fact 4: From *Fact 2*, $X_n(a)\geq X_1(a)+\frac{(n-1)\kappa}{\lambda_b d_b(w)}$. If $N(a)$ is infinite, then when $n\to\infty$, $X_n(a)\geq X_{n-1}(a)=+\infty$, which will lead to $R_n<0$, in contradiction with the condition that the algorithm stops. $\square$

LEMMA EC.1. *Under Assumption 1 and 2, when $f^*\geq f^{**}>0$, there exists a unique $a^*\in(0,f^{**}]$ such that when the input is a^*, Algorithm 1 outputs $R_N=0$, $N<\infty$, $X_0\leq 0$ and $X_i\geq 0$ for $i>0$. That is, one solution to (8) can be found by setting $M=N(a^*)$, $f_1=X_N(a^*)$, $\cdots$, $f_M=X_1(a^*)$, $f_{M+1}=X_0(a^*)$.*

Proof of Lemma EC.1: We divide the proof into several steps.

Increase the input $a\to a+\Delta a$ by an infinitesimal positive amount Δa. Denote $\tilde{N}=\min\{N(a),N(a+\Delta a)\}$ and $X_n(a+\Delta a)=X_n(a)+\Delta X_n$ for $n\leq\tilde{N}$. With a slight abuse of notation, we use X_n to refer to $X_n(a)$ below.

Step (1): $\Delta X_{\tilde{N}}>\cdots>\Delta X_n>\cdots>\Delta X_1>\Delta X_0>0$.

Note that $X_0 = X_1 - \frac{\kappa}{\lambda_b d_b(X_1+w)}$ and $X_0 + \Delta X_0 = X_1 + \Delta X_1 - \frac{\kappa}{\lambda_b d_b(X_1+\Delta X_1+w)}$ from Step 3 in the algorithm when the input is a and $a+\Delta a$ respectively. These equalities can be rewritten as $\lambda_b d_b(X_1+w)(X_1-X_0)=\kappa$ and $\lambda_b d_b(X_1+\Delta X_1+w)((X_1+\Delta X_1)-(X_0+\Delta X_0))=\kappa$. Therefore

$$\begin{aligned}0 =&\lambda_b d_b(X_1+\Delta X_1+w)((X_1+\Delta X_1)-(X_0+\Delta X_0))-\lambda_b d_b(X_1+w)(X_1-X_0)\\ =&\frac{d\lambda_b d_b(x+w)(x-X_0)}{dx}\,|_{x=X_1}\Delta X_1-\lambda_b d_b(X_1+w)\Delta X_0\end{aligned}$$

where the second equality follows from a Taylor expansion. From *Fact 1*, $\frac{d\lambda_b d_b(x+w)(x-X_0)}{dx}\,|_{x=X_1}>0$. Then due to $\Delta X_1=\Delta a>0$, the equation above will lead to $\Delta X_0>0$. Besides, the equation above can be rewritten as

$$\lambda_b d_b'(X_1+w)(X_1-X_0)\Delta X_1+\lambda_b d_b(X_1+w)(\Delta X_1-\Delta X_0)=0.$$

From *Fact 1*, $X_1>X_0$. As $\lambda_b d_b'(X_1+w)<0$ and $\Delta X_1>0$, the first term in the left hand above is negative, and thus the second term in the left hand above is positive, which leads to $\Delta X_1>\Delta X_0>0$.

Now from $\Delta X_1>\Delta X_0$ and $X_1>X_0$, one can show that $(X_1+\Delta X_1)-(X_0+\Delta X_0)>X_1-X_0>0$, $(X_1+\Delta X_1)-X_1>(X_0+\Delta X_0)-X_0>0$. Then from Lemma EC.4 in Section EC.8, by setting $y_2=X_1+\Delta X_1$, $z_2=X_0+\Delta X_0$, $y_1=X_1$ and $z_1=X_0$, we have $\Delta X_2-\Delta X_1>0$. By induction, for all $1\le n\le \tilde{N}$, $\Delta X_n>\Delta X_{n-1}>\cdots>\Delta X_1>0$.

Step (2): $N(a+\Delta a)\le N(a)$ and $R_n(a+\Delta a)<R_n(a)$ for all $1\le n\le N(a+\Delta a)$.

By the definition in the algorithm $R_n(a)=\mathcal{R}_b(X_n(a))-\kappa-\lambda_s d_s(X_{n-1}(a))(X_n(a)-X_{n-1}(a))$. Then for all $1\le n\le\tilde{N}$, by Taylor's expansion, $R_n(a+\Delta a)=R_n(a)+\mathcal{R}_b'(X_n+w)\Delta X_n-[\lambda_s d_s(X_{n-1})(\Delta X_n-\Delta X_{n-1})+\lambda_s d_s'(X_{n-1})(X_n-X_{n-1})\Delta X_{n-1}]$. As $X_n\ge X_{n-1}$ by *Fact 2* and $\Delta X_n\ge\Delta X_{n-1}>0$ by Step (1), we have $R_n(a+\Delta a)<R_n(a)$.

Next, we prove $N(a+\Delta a)\le N(a)$ by contradiction. Suppose $N(a+\Delta a)>N(a)$. Then by the condition that the algorithm stops, $R_{N(a)}(a+\Delta a)>0$ and $R_{N(a)}(a)\le 0$, contradicting $R_{N(a)}(a+\Delta a)<R_{N(a)}(a)$ we have just proved. As a result, $N(a+\Delta a)\le N(a)$.

Step (3): There exists a unique value, $a^*\in(0,f^{**}]$, such that $R_{N(a^*)}(a^*)=0$, $N(a^*)<\infty$, $X_0(a^*)\le 0$ and $X_i(a^*)\ge 0$ for $i>0$.

From Step (2), $N(a)$ is non-increasing in a. From *Fact 3*, $N(f^{**})=N(0)-1$. As a result, $N(0)\ge N(a)\ge N(f^{**})=N(0)-1$ when $a\in(0,f^{**}]$. That is, when $a\in(0,f^{**}]$, the algorithm terminates with $N(a)\in\{N(0)-1,N(0)\}$. Next we show that $N(a^*)=N(0)-1$. Suppose that the algorithm terminates with $N=N(0)$. From Step (2), the termination of the algorithm must satisfy $R_N<R_{N(0)}(0)\le 0$. As a result, if the output of algorithm satisfies $R_{N(a^*)}(a^*)=0$, we must have $N(a^*)=N(0)-1$. From *Fact 3*, $R_{N(0)-1}(f^{**})\le 0$ and $R_{N(0)-1}(0)>0$. Then because of Step

(2), $R_{N(0)-1}(a)$ is decreasing in a and thus there exists a unique value $a^* \in (0, f^{**}]$ such that $R_{N(a^*)}(a^*) = 0$. By *Fact 1*, *Fact 2* and *Fact 4*, $N(a^*) < \infty$, $X_0(a^*) \le 0$ and $X_i(a^*) \ge 0$ for $i > 0$.

By the discussion of the intuition to design the algorithm, one solution to (8) can be derived by setting $M = N(a^*)$, $f_1 = X_N(a^*), \cdots, f_M = X_1(a^*)$, $f_{M+1} = X_0(a^*)$. $\square$

EC.3.1.3. Structural Properties In this subsection, we show the structural properties of any solution to (8), using the definition of the last section.

LEMMA EC.2. *Under Assumption 1 and 2, any solution to* (8) *satisfies the following properties, (1) f_m is decreasing and convex in m. (2) For $1 < m \le M+1$, we have $\frac{d}{dx}[\lambda_b d_b(x+w)(x - f_m)]|_{x=f_{m-1}} > 0$, and $\frac{\lambda_b d_b(x+w)x + \lambda_s d_s(f_{m+1})f_{m+1} - \kappa}{\lambda_b d_b(x+w) + \lambda_s d_s(f_{m+1})} < f_m$ when $x < f_{m-1}$. (3) $f_1 \le f^*$ and $f_M \le f^{**}$. (4) $M = 0$ if and only if $f^* \le 0$; $M = 1$ if and only if $0 < f^* \le f^{**}$; $M > 1$ if and only if $f^* > f^{**}$.*

Proof of Lemma EC.2: To show part (1), we first show f_m is decreasing in m. For $m \ge M$, according to the m-th equation in (8), $f_{m-1} - f_m = \frac{\kappa}{\lambda_b d_b(f_{m-1}+w)} > 0$. Suppose there exists $m < M$ such that $f_{m-1} - f_m \le 0$. Then according to the mth equation in (8), $\lambda_s d_s(f_{m+1})(f_m - f_{m+1}) + \kappa = \lambda_b d_b(f_{m-1}+w)(f_{m-1} - f_m) \le 0$. Therefore, $f_m \le f_{m+1}$. By induction, from the mth to the $(M-1)$-th equations in (8), we have $f_{M-1} \le f_M$. However, from the M-th equation in (8), $f_{M-1} - f_M = \frac{\kappa}{\lambda_b d_b(f_{M-1}+w)} > 0$. This leads to a contradiction and proves that f_m must be decreasing.

Next we show that $f_m - f_{m+1}$ is decreasing in m in part (1). For $m \ge M$, $f_m - f_{m+1} = \frac{\kappa}{\lambda_b d_b(f_m+w)}$, which is clearly decreasing in m as f_m is decreasing in m and $d_b(\cdot)$ is a decreasing function on its support. For $m < M$, starting from the $(M-1)$-th and Mth equations in (8),

$$\begin{aligned}\lambda_s d_s(f_M)(f_{M-1} - f_M) + \kappa &= \lambda_b d_b(f_{M-2}+w)(f_{M-2} - f_{M-1}),\\ \kappa &= \lambda_b d_b(f_{M-1}+w)(f_{M-1} - f_M).\end{aligned}$$

We have $\lambda_b d_b(f_{M-2}+w)(f_{M-1} - f_M) < \lambda_b d_b(f_{M-1}+w)(f_{M-1} - f_M) = \kappa < \lambda_b d_b(f_{M-2}+w)(f_{M-2} - f_{M-1})$, where the first inequality follows from $f_{M-2} > f_{M-1}$, the equality follows from the second equation above, and the second inequality follows from the first equation above and $f_{M-1} \ge f_M$. As a result, $f_{M-1} - f_M < f_{M-2} - f_{M-1}$. Suppose $f_{m-1} - f_m < f_{m-2} - f_{m-1}$ for some $m < M$. Then

$$\begin{aligned}\lambda_b d_b(f_{m-2}+w)(f_{m-2} - f_{m-1}) &= \lambda_s d_s(f_m)(f_{m-1} - f_m) + \kappa\\ &< \lambda_s d_s(f_m)(f_{m-2} - f_{m-1}) + \kappa < \lambda_s d_s(f_{m-1})(f_{m-2} - f_{m-1}) + \kappa\\ &= \lambda_b d_b(f_{m-3}+w)(f_{m-3} - f_{m-2}) < \lambda_b d_b(f_{m-2}+w)(f_{m-3} - f_{m-2}),\end{aligned}$$

where the equalities follow from (8), the first inequality follows from $f_{m-1} - f_m < f_{m-2} - f_{m-1}$, the second inequality follows from $f_{m-1} > f_m$ and the last inequality follows from $f_{m-2} < f_{m-3}$. As a result, $f_{m-2} - f_{m-1} < f_{m-3} - f_{m-2}$. By induction, we have shown the claim.

Next we show part (2): $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_m)]|_{x=f_{m-1}} > 0$ for $1 < m \leq M$. Note that

$$\begin{aligned}\lambda_b d_b(f_{m-1}+w)(f_{m-1}-f_m) =& \kappa + \lambda_s d_s(f_{m+1})(f_m - f_{m+1}) < \kappa + \lambda_s d_s(f_m)(f_{m-1}-f_m)\\ =& \lambda_b d_b(f_{m-2}+w)(f_{m-2}-f_{m-1}) < \lambda_b d_b(f_{m-2}+w)(f_{m-2}-f_m),\end{aligned}$$

where the equalities follow from (8), the first inequality follows from $f_{m+1} < f_m$ and $f_m - f_{m+1} < f_{m-1} - f_m$, and the last inequality is due to $f_m < f_{m-1}$. Let $x^* = \arg\max_x d_b(x+w)(x-f_m)$. By Assumption 1, $d_b(x+w)(x-f_m)$ is unimodal and increasing in $x \in [0, x^*)$. Since $f_{m-1} < f_{m-2}$, the inequality above indicates that $f_{m-1} < x^*$. Thus, $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_m)]|_{x=f_{m-1}} > 0$. Moreover, when $x < f_{m-1}$, we have

$$\begin{aligned}&\frac{\lambda_b d_b(x+w)x + \lambda_s d_s(f_{m+1})f_{m+1} - \kappa}{\lambda_b d_b(x+w) + \lambda_s d_s(f_{m+1})} - f_m\\ =&\frac{\lambda_b d_b(x+w)(x-f_m) + \lambda_s d_s(f_{m+1})(f_{m+1}-f_m) - \kappa}{\lambda_b d_b(x+w) + \lambda_s d_s(f_{m+1})}\\ <&\frac{\lambda_b d_b(f_{m-1}+w)(f_{m-1}-f_m) + \lambda_s d_s(f_{m+1})(f_{m+1}-f_m) - \kappa}{\lambda_b d_b(x+w) + \lambda_s d_s(f_{m+1})}\\ =&0,\end{aligned}$$

where the inequality is because $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_m)] > 0$ when $x < f_{m-1}$, and the last equality is due to (8).

Next, we show part (3): $f_1 \leq f^*$ and $f_M \leq f^{**}$. Note that $\max_p\{\lambda_b d_b(p+w)(p-f_1)\} = \lambda_s d_s(f_2)(f_1-f_2) + \kappa \geq \kappa = \max_p\{\lambda_b d_b(p+w)(p-f^*)\}$, where the first equality is due to (8) and the second equality is due to the definition of f^*, and the inequality is due to the fact that f_m is decreasing in m. As $\max_p\{\lambda_b d_b(p+w)(p-z)\}$ is decreasing in z, $f_1 \leq f^*$. To show $f_M \leq f^{**}$, we only need to focus on the case where f^{**} is finite. According to the definition of f^{**}, $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f^{**}} \geq 0$. Besides,

$$\lambda_b d_b(f^{**}+w)f^{**} = \kappa = \lambda_b d_b(f_M+w)(f_M - f_{M+1}) \geq \lambda_b d_b(f_M+w)f_M,$$

where the first equality is due to the definition of f^{**}, the second equality is due to the $(M+1)$th equation in (8), and the inequality is due to $f_{M+1} \leq 0$. Also, since $\frac{d}{dx}[\lambda_b d_b(x+w)(x-f_{M+1})]|_{x=f_M} \geq 0$ by part (2), we have $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f_M} \geq 0$. As $\lambda_b d_b(x+w)x$ is unimodal by Assumption 1, the inequality above indicates $f_M \leq f^{**}$.

Finally, we show part (4): $M=0$ if and only if $f^* \leq 0$; $M=1$ if and only if $0 < f^* \leq f^{**}$; $M>1$ if and only if $f^* > f^{**}$.

We first show that they are *necessary conditions*. a) If $M=0$, $f_1 = f^* \leq 0$. b) If $M=1$, due to the monotonicity of f_m, it requires that $f_0 > f_1 = f^* > 0 > f_2$. Then $\lambda_b d_b(f_1+w)f_1 \leq \kappa = \lambda_b d_b(f_0 + w)(f_0 - f_1) < \lambda_b d_b(f_0+w)f_0$, where the first inequality is due to $f_2 = f_1 - \frac{\kappa}{\lambda_b d_b(f_1+w)} \leq 0$, the equality is due to the first equation in (8), and the last inequality is due to $f_1 > 0$. By Assumption 1, $\lambda_b d_b(p+w)p$ is unimodal. As $f_1 < f_0$, the inequality above leads to $\frac{d}{dp}[\lambda_b d_b(p+w)p]|_{p=f_1} \geq 0$. Besides, $\lambda_b d_b(f^{**}+w)f^{**} = \kappa \geq \lambda_b d_b(f_1+w)f_1$, where the equality is due to the definition of f^{**} and the inequality is due to $f_2 = f_1 - \frac{\kappa}{\lambda_b d_b(f_1+w)} \leq 0$. Then we have $f^{**} \geq f_1 = f^*$.

c) If $M > 1$, we know $\lambda_b d_b(f_1+w)(f_1 - f_2) = \kappa + \lambda_s d_s(f_3)(f_2 - f_3) \geq \kappa$, where the equality is due to (8) and the inequality is due to $f_2 > f_3$. As $f_2 > 0$ for $M > 1$, it leads to $\lambda_b d_b(f_1+w)f_1 > \kappa$. Then by the definition of f^{**}, $\lambda_b d_b(f_1+w)f_1 > \lambda_b d_b(f^{**}+w)f^{**}$. As $\frac{d}{dx}[\lambda_b d_b(x+w)x]|_{x=f^{**}} \geq 0$ and $\lambda_b d_b(x+w)x$ is unimodal in x, we have $f_1 > f^{**}$. Thus by part (3) $f^* > f^{**}$.

Next we show they are *sufficient conditions*. a) For $f^* < 0$, since $f_1 \leq f^* \leq 0$ by part (3), we have $M = 0$. b) For $0 < f^* \leq f^{**}$, suppose $M > 1$, we must have $f^* \geq f_1 > f^{**}$ from the necessary condition, leading to contradiction. Suppose $M = 0$, then $f_1 = f^* \leq 0$, leading to contradiction. Therefore, $M = 1$. c) For $f^* > f^{**}$, suppose $M \leq 1$, we must have $f^* \leq 0$ or $0 < f^* \leq f^{**}$ from the necessary condition, leading to contradiction. Therefore, $M > 1$. □

Proof of Theorem 1: We establish the existence in Section EC.3.1.1 and Section EC.3.1.2. We then need the following steps to show the uniqueness.

Step (1): The solution to (8) is unique when $f^* \leq 0$ or $0 < f^* \leq f^{**}$. By Lemma EC.2, $M \leq 1$. Then the unique solution to (8) can be directly solved, having the form in (EC.5).

Step (2): The solution to (8) is unique when $f^* > f^{**}$. We will show that any solution can be derived from the proposed algorithm with the output $R_N = 0$. Let $[f_1, \cdots, f_M, \cdots]$ be a solution to (8). By part (2) in Lemma EC.2, for $1 < m \leq M$, $\frac{dR(x; f_m, f_{m+1})}{dx}|_{x=f_{m-1}} = \frac{d}{dx}[\lambda_b d_b(x+w)(x - f_m)]|_{x=f_{m-1}} > 0$. Due to (8) , for $1 < m \leq M$, $R(f_{m-1}; f_m, f_{m+1}) = 0$, then $\sup_x R(x; f_m, f_{m+1}) > 0$. Besides, the first equation in (8) can be rewritten as $\sup_x R(x; f_1, f_2) = 0$. Therefore if we set the input a to be f_M in the solution, then the output of the algorithm satisfies $R_N = 0$, $N = M$, $X_0 = f_{M+1}$, $X_1 = f_M$, $\cdots$, $X_N = f_1$. Due to $f_M \leq f^{**}$ by part (3) in Lemma EC.2 and Lemma EC.1, the solution to (8) is uniquely output by Algorithm 1. This proves Theorem 1. □

Proof of Proposition 3 It is straightforward to prove part one and part two by part one in Lemma EC.2.

For part three, suppose M is unchanged when we change κ. We will examine how f_m changes with respect to κ. Taking derivative with respect to κ on both sides of the recursive system (8) yields,

$$\begin{aligned} \lambda_s d_s(f_2)(f_1' - f_2') + a_1 f_2' + 1 &= b_1 f_1', \\ \lambda_s d_s(f_{m+1})(f_m' - f_{m+1}') + a_m f_{m+1}' + 1 &= b_m f_{m-1}' + \lambda_b d_b(f_{m-1}+w)(f_{m-1}' - f_m') \; M-1 \geq m \geq 2, \\ 1 &= b_M f_{M-1}' + \lambda_b d_b(f_{M-1}+w)(f_{M-1}' - f_M'), \end{aligned} \tag{EC.6}$$

where $f_i' = \frac{df_i}{d\kappa}$, $a_m = \lambda_s d_s'(f_{m+1})(f_m - f_{m+1})$, and

$$b_m = \begin{cases} \frac{d}{df_1}[\max_{f_0} \{\lambda_b d_b(f_0 + w)(f_0 - f_1)\}], & m = 1, \\ \lambda_b d_b'(f_{m-1} + w)(f_{m-1} - f_m), & m > 1. \end{cases}$$

From the properties of $d_b(\cdot)$, $d_s(\cdot)$ and the monotonicity of f_m, we have $a_m > 0$, $b_m < 0$ for any m. As $\frac{d}{dx}[\lambda_b d_b(x + w)(x - f_m)]|_{x=f_{m-1}} > 0$ by part (2) in Lemma EC.2, rewriting this inequality, we have $b_m + \lambda_b d_b(f_{m-1} + w) > 0$.

Next, we prove $f_M' < 0$ by contradiction. Suppose $f_M' \geq 0$. From the last equation in (EC.6), we have $1 = (b_M + \lambda_b d_b(f_{M-1} + w))f_{M-1}' - \lambda_b d_b(f_{M-1} + w)f_M'$. As $b_M + \lambda_b d_b(f_{M-1} + w) > 0$, we must have $f_{M-1}' > 0$. Then as $b_M < 0$ and $f_{M-1}' > 0$, this equation also shows $1 = b_M f_{M-1}' + \lambda_b d_b(f_{M-1} + w)(f_{M-1}' - f_M') \leq \lambda_b d_b(f_{M-1} + w)(f_{M-1}' - f_M')$. Therefore $f_{M-1}' > f_M' \geq 0$. By recursively applying the same argument to the other equations in (EC.6) (replacing the left-hand side 1 by $\lambda_s d_s(f_{m+1})(f_m' - f_{m+1}') + a_m f_{m+1}' + 1$, which is still positive), we have $f_1' > f_2' > 0$. This cannot happen, as in the first equation of (EC.6), if $f_1' > f_2' > 0$, as $a_1 > 0$, the left hand is positive; however, as $b_1 < 0$, the right hand is negative, leading to a contradiction. As a result, $f_M' < 0$.

The above proof shows that when κ increases, f_M decreases. When f_M becomes zero, then the endogenous capacity M decreases by one. Therefore, M is decreasing in κ. Similarly we can prove how f_M changes in λ_b and λ_s. The endogenous capacity is decreasing (increasing, decreasing, decreasing) in κ (λ_b, λ_s, w).

Due to part four in Lemma EC.2, $M \geq 1$ if and only if $\max_p\{d_b(p + w)p\} \geq \kappa/\lambda_b$.

For part four, we divide the proof into several steps.

(1) $V_{\mathcal{M}}(m)$ is increasing in m. From the first equation in (9), $V_{\mathcal{M}}(0) < V_{\mathcal{M}}(1)$. We prove the claim for $m > 0$ by contradiction. Suppose $V_{\mathcal{M}}(l - 1) < V_{\mathcal{M}}(l)$ holds for $l \leq m$, but $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m + 1)$ for some $m \geq 1$. Then we have

$$\begin{aligned} &[r + \lambda_s d_s(f_{m+1}) + \lambda_b d_b(f_{m-1} + w)]V_{\mathcal{M}}(m) \\ =&\lambda_s d_s(f_{m+1})V_{\mathcal{M}}(m + 1) + \lambda_b d_b(f_{m-1} + w)[V_{\mathcal{M}}(m - 1) + w] \\ <&\lambda_s d_s(f_{m+1})V_{\mathcal{M}}(m) + \lambda_b d_b(f_{m-1} + w)[V_{\mathcal{M}}(m) + w], \end{aligned}$$

where the equality is due to (9), and the inequality is due to our induction hypothesis that $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m + 1)$ and $V_{\mathcal{M}}(m - 1) \leq V_{\mathcal{M}}(m)$. After arranging the terms, we have $rV_{\mathcal{M}}(m) < \lambda_b d_b(f_{m-1} + w)w$. Due to the monotonicity of f_m, $rV_{\mathcal{M}}(m) < \lambda_b d_b(f_m + w)w$. Since $V_{\mathcal{M}}(m) \geq V_{\mathcal{M}}(m + 1)$, we have $rV_{\mathcal{M}}(m + 1) < \lambda_b d_b(f_m + w)w$. As a result,

$$\begin{aligned} &(\lambda_s d_s(f_{m+2}) + \lambda_b d_b(f_m + w))V_{\mathcal{M}}(m + 1) + \lambda_b d_b(f_m + w)w \\ >&(\lambda_s d_s(f_{m+2}) + \lambda_b d_b(f_m + w) + r)V_{\mathcal{M}}(m + 1) \\ =&\lambda_s d_s(f_{m+2})V_{\mathcal{M}}(m + 2) + \lambda_b d_b(f_m + w)[V_{\mathcal{M}}(m) + w] \\ \geq&\lambda_s d_s(f_{m+2})V_{\mathcal{M}}(m + 2) + \lambda_b d_b(f_m + w)(V_{\mathcal{M}}(m + 1) + w), \end{aligned}$$

where the equality is due to (9), the first inequality is due to $rV_{\mathcal{M}}(m+1) < \lambda_b d_b(f_m + w)w$, the second inequality is due to $V_{\mathcal{M}}(m+1) \le V_{\mathcal{M}}(m)$. Arranging the terms, we have $V_{\mathcal{M}}(m+1) \ge V_{\mathcal{M}}(m+2)$. Moreover, as $rV_{\mathcal{M}}(m+1) < \lambda_b d_b(f_m + w)w$ and $f_m > f_{m+1}$, we have $rV_{\mathcal{M}}(m+2) < \lambda_b d_b(f_m + w)w < \lambda_b d_b(f_{m+1} + w)w$. By induction, we can show $V_{\mathcal{M}}(M-1) \ge V_{\mathcal{M}}(M)$ and $rV_{\mathcal{M}}(M) < \lambda_b d_b(f_{M-1} + w)w$. However, this leads to contradiction as the last equation in (9) can not hold. Thus, $V_{\mathcal{M}}(m)$ is increasing in m.

(2) $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ is decreasing in m. By the last two equations of 9, we have

$$\begin{aligned}
&\lambda_b d_b(f_{M-2}+w)[V_{\mathcal{M}}(M-1)-V_{\mathcal{M}}(M-2)-w]\\
=&-rV_{\mathcal{M}}(M-1)+\lambda_s d_s(f_M)[V_{\mathcal{M}}(M)-V_{\mathcal{M}}(M-1)]\\
>&-rV_{\mathcal{M}}(M)=\lambda_b d_b(f_{M-1}+w)[V_{\mathcal{M}}(M)-V_{\mathcal{M}}(M-1)-w]
\end{aligned}$$

where the inequality is due to $V_{\mathcal{M}}(M) > V_{\mathcal{M}}(M-1)$. Since f_m is decreasing, $\lambda_b d_b(f_{M-1}+w) > \lambda_b d_b(f_{M-2}+w)$, and the inequality above leads to $V_{\mathcal{M}}(M-1) - V_{\mathcal{M}}(M-2) > V_{\mathcal{M}}(M) - V_{\mathcal{M}}(M-1)$. Now we use induction. Suppose $V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2) > V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ for some $m \le M$. Then

$$\begin{array}{rl}
&\lambda_b d_b(f_{m-3}+w)[V_{\mathcal{M}}(m-2)-V_{\mathcal{M}}(m-3)-w]\\
=&-rV_{\mathcal{M}}(m-2)+\lambda_s d_s(f_{m-1})[V_{\mathcal{M}}(m-1)-V_{\mathcal{M}}(m-2)]\\
>&-rV_{\mathcal{M}}(m-1)+\lambda_s d_s(f_m)[V_{\mathcal{M}}(m)-V_{\mathcal{M}}(m-1)]\\
=&\lambda_b d_b(f_{m-2}+w)[V_{\mathcal{M}}(m-1)-V_{\mathcal{M}}(m-2)-w]
\end{array},$$

where we have used $\lambda_s d_s(f_{m-1}) \ge \lambda_s d_s(f_m)$ and $V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2) > V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1)$ in the inequality and the equalities are due to (9). Similarly, as $d_b(f_{m-2}+w) > d_b(f_{m-3}+w)$, we have $V_{\mathcal{M}}(m-2) - V_{\mathcal{M}}(m-3) > V_{\mathcal{M}}(m-1) - V_{\mathcal{M}}(m-2)$. This completes the proof.

(3) $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1) \le w$. As f_m is decreasing in m, the arrival rate of the sellers is bounded by $\lambda_s d_s(f_1)$, then we must have that for any m, $V_{\mathcal{M}}(m) \le \frac{\lambda_s d_s(f_1)w}{r}$. We then prove the claim by contradiction.

Suppose $V_{\mathcal{M}}(m) - V_{\mathcal{M}}(m-1) > w$ for some m. Then due to the concavity of $V_{\mathcal{M}}(m)$, we have $V_{\mathcal{M}}(1) - V_{\mathcal{M}}(0) > w$. As $(r + \lambda_s d_s(f_1))V_{\mathcal{M}}(0) = \lambda_s d_s(f_1)V_{\mathcal{M}}(1)$ from (9), then $(r + \lambda_s d_s(f_1))V_{\mathcal{M}}(0) > \lambda_s d_s(f_1)[V_{\mathcal{M}}(0) + w]$. That is $V_{\mathcal{M}}(0) > \frac{\lambda_s d_s(f_1)w}{r}$. This leads to a contradiction. $\square$

EC.3.2. Proofs for Section 2.2.2

Proof of Proposition 4: For part (1), from (8) , we have

$$d_s(f^{(n)}_{M^*+2})(f^{(n)}_{M^*+1} - f^{(n)}_{M^*+2}) + \frac{\kappa}{n\lambda_s} = \frac{\lambda_b}{\lambda_s} d_b(f^{(n)}_{M^*} + w)(f^{(n)}_{M^*} - f^{(n)}_{M^*+1}) > \frac{\lambda_b}{\lambda_s} d_b(f^{(n)}_{M^*} + w)(f^{(n)}_{M^*+1} - f^{(n)}_{M^*+2}).$$

Arranging the terms of the two sides of the inequality, we have $\lim_{n\to\infty}(\frac{\lambda_b}{\lambda_s} d_b(f^{(n)}_{M^*} + w) - d_s(f^{(n)}_{M^*+2}))((f^{(n)}_{M^*+1} - f^{(n)}_{M^*+2})) \le 0$. We then prove $\lim_{n\to\infty}(f^{(n)}_{M^*+1} - f^{(n)}_{M^*+2}) = 0$ by contradiction. If for a subsequence $n_k \to \infty$, $\lim_{k\to\infty}(f^{(n_k)}_{M^*+1} - f^{(n_k)}_{M^*+2}) > 0$, to make the equation above hold, we have $\lim_{k\to\infty}\left(\frac{\lambda_b}{\lambda_s}d_b(f_{M^*}^{(n_k)}+w)-d_s(f_{M^*+2}^{(n_k)})\right)\leq 0$. However, from the definition of M^*, we have $\frac{\lambda_b}{\lambda_s}d_b(f_{M^*}^{(n_k)}+w)\geq d_s(f_{M^*+1}^{(n_k)})\geq d_s(f_{M^*+2}^{(n_k)})$. Then we must have $\lim_{k\to\infty}(\frac{\lambda_b}{\lambda_s}d_b(f_{M^*}^{(n_k)}+w)-d_s(f_{M^*+2}^{(n_k)}))=0$. As a result, $\lim_{k\to\infty}(d_s(f_{M^*+1}^{(n_k)})-d_s(f_{M^*+2}^{(n_k)}))=0$ and thus $\lim_{k\to\infty}(f_{M^*+1}^{(n_k)}-f_{M^*+2}^{(n_k)})=0$, leading to contradiction. Therefore, $\lim_{n\to\infty}(f_{M^*+1}^{(n)}-f_{M^*+2}^{(n)})=0$. Then using (8), by taking the limit of $n\to\infty$, the waiting cost is negligible and one can easily prove that for any finite k, $\lim_{n\to\infty}[f_{M^*\pm k}^{(n)}-f_{M^*}^{(n)}]=0$. As $f_{M^*+1}^{(n)}\leq\hat{f}$ and $f_{M^*-1}^{(n)}\geq\hat{f}$, $\lim_{n\to\infty}f_{M^*\pm k}^{(n)}=\hat{f}$.

For part (2), we need several steps to complete the proof.

1) $\lim_{n\to\infty}M^{*(n)}=+\infty$. From the first equation in (8), we have $\lambda_s d_s(f_2^{(n)})(f_1^{(n)}-f_2^{(n)})+\frac{\kappa}{n}=\max_{f_0}\{\lambda_b d_b(f_0+w)(f_0-f_1^{(n)})\}$. If the claim does not hold, then from part (1), we have $\lim_{n\to\infty}f_1^{(n)}=\lim_{n\to\infty}f_2^{(n)}=\hat{f}$ (for a subsequence of n with a slight abuse of notation), then the left-hand side of the equation above is 0, whereas the right-hand side is positive, which leads to a contradiction.

2) For any positive ϵ, $\lim_{n\to\infty}\pi_{n_1}=\lim_{n\to\infty}\pi_{n_2}=0$, where $n_1=\max\{m:f_m^{(n)}>\hat{f}+\epsilon\}$ and $n_2=\min\{m:f_m^{(n)}<\hat{f}-\epsilon\}$. Recall that π_m is the steady-state distribution of state m, which depends on n as well. Clearly, $n_1<M^*$. Since $\lim_{n\to\infty}f_{M^*\pm k}^{(n)}=\hat{f}$ for finite k, $\lim_{n\to\infty}(M^*-n_1)=\infty$. Besides, for any $m\in[n_1,M^*]$, we know $\pi_m\geq\pi_{n_1}$, because of the definition of π_m in (10), the monotonicity of f_m and the definition of M^*. Then, we have $1\geq\lim_{n\to\infty}\sum_{m=n_1}^{M^*}\pi_m\geq\lim_{n\to\infty}\sum_{m=n_1}^{M^*}\pi_{n_1}=\lim_{n\to\infty}(M^*-n_1)\pi_{n_1}$. As $\lim_{n\to\infty}(M^*-n_1)=\infty$, $\lim_{n\to\infty}\pi_{n_1}=0$. Similarly, $\lim_{n\to\infty}\pi_{n_2}=0$.

3) $\lim_{n\to\infty}\sum_{m:|f_m^{(n)}-\hat{f}|>\epsilon}\pi_m=0$. For $m\in[0,n_1]$, because of (10) and $f_m\leq f_{m-1}$, we have $\pi_{m-1}<a\pi_m$, where $a_n\leq\frac{\lambda_b d_b(f_{n_1-1}^{(n)}+w)}{\lambda_s d_s(f_{n_1+1}^{(n)})}$. As a result, we have $\lim_{n\to\infty}\sum_{m:f_m^{(n)}>\hat{f}+\epsilon}\pi_m\leq\lim_{n\to\infty}\sum_{n=0}^{\infty}a_n^n\pi_{n_1}=\lim_{n\to\infty}\frac{1}{1-a_n}\pi_{n_1}$. As $\lim_{n\to\infty}\frac{\lambda_b d_b(f_{n_1-1}^{(n)}+w)}{\lambda_s d_s(f_{n_1}^{(n)})}<\lim_{n\to\infty}\frac{\lambda_b d_b(f_{M^*}^{(n)}+w)}{\lambda_s d_s(\hat{f}+\epsilon)}=\frac{\lambda_b d_b(\hat{f}+w)}{\lambda_s d_s(\hat{f}+\epsilon)}<\frac{\lambda_b d_b(\hat{f}+w)}{\lambda_s d_s(\hat{f})}=1$, we have $a_n<1-\delta$ being bounded away from 1 and thus $\lim_{n\to\infty}\sum_{m:f_m^{(n)}>\hat{f}+\epsilon}\pi_m=0$ because of step 2) above. Similarly, $\lim_{n\to\infty}\sum_{m:f_m^{(n)}<\hat{f}-\epsilon}\pi_m=0$. Thus we complete the proof. □

Proposition EC.2. *Given $w=c_s+c_b$, in the thick market, the firm's long-run average revenue converges:* $\lim_{n\to\infty}\frac{V_{\mathcal{M}}^{(n)}}{n}=\frac{\lambda_s d_s(\hat{f})w}{r}$.

Proof of Proposition EC.2: For any positive ϵ, define $n_1=\max\{m:f_m^{(n)}>\hat{f}+\epsilon\}$ and $n_2=\min\{m:f_m^{(n)}<\hat{f}-\epsilon\}$. Then

$$
\begin{aligned}
\frac{rV_{\mathcal{M}}^{(n)}}{nw}&=\sum_{m=1}^{M^{(n)}}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w)\\
&=\sum_{m=1}^{n_1}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w)+\sum_{m=n_1+1}^{n_2-1}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w)+\sum_{m=n_2}^{M^{(n)}}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w).
\end{aligned}
$$

For the first term,

$$
\lim_{n\to\infty}\sum_{m=1}^{n_1}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w)\leq\lim_{n\to\infty}\sum_{m=1}^{n_1}\pi_m\lambda_b d_b(f_{n_1-1}^{(n)}+w)\leq\lambda_b d_b(\hat{f}+\epsilon+w)\lim_{n\to\infty}\sum_{m=1}^{n_1}\pi_m=0,
$$

where we have used $\lim_{n\to\infty}\sum_{m=1}^{n_1}\pi_m=0$ from Proposition 4. Similarly, the last term is also zero when $n\to\infty$. For the second term, as for any $m\in[n_1+1,n_2-1]$, $d_b(\hat{f}+\epsilon+w)\le d_b(f_{m-1}^{(n)}+w)\le d_b(\hat{f}-\epsilon+w)$, we have

$$\lambda_b d_b(\hat{f}+\epsilon+w)\lim_{n\to\infty}\sum_{m=n_1+1}^{n_2-1}\pi_m\le\lim_{n\to\infty}\sum_{m=n_1+1}^{n_2-1}\pi_m\lambda_b d_b(f_{m-1}^{(n)}+w)\le\lambda_b d_b(\hat{f}-\epsilon+w)\lim_{n\to\infty}\sum_{m=n_1+1}^{n_2-1}\pi_m.$$

From Proposition 4, $\lim_{n\to\infty}\sum_{m=n_1+1}^{n_2-1}\pi_m=1$. Therefore,

$$\lambda_b d_b(\hat{f}+\epsilon+w)\le\lim_{n\to\infty}\frac{rV_{\mathcal{M}}^{(n)}}{nw}\le\lambda_b d_b(\hat{f}-\epsilon+w).$$

Since it holds for an arbitrary $\epsilon>0$, we have completed the proof. $\square$

EC.4. Proofs for Section 3

Proof of Theorem 2: By Lemma 1, we have that the maximal number of sellers can be bounded, i.e., $M\le\frac{2\bar{p}\lambda_b}{\kappa}$. When the market size of buyers is relatively small ($\lambda_b<\frac{\kappa}{2\bar{p}}$) or the waiting cost κ is relatively high ($\kappa>2\bar{p}\lambda_b$), the cost of selling a product on the marketplace is too large such that no seller will join the marketplace, i.e., $M=0$ and $V_{\mathcal{M}}=0$. In contrast, when $d_s(p)>0$ for any $p>0$, the dealer can always achieve a positive expected profit, $V_{\mathcal{D}}(0)>0$, by setting

$$p_b(0)=\infty,\ p_b(x)=\eta\quad\forall\, x\ge 1,\ p_s(0)=e^{-\frac{r}{\lambda_b d_b(\eta)}}\eta-\epsilon,\ p_s(x)=0\quad\forall\, x\ge 1,$$

where η is a positive value such that $d_b(\eta)>0$ and $d_s(e^{-\frac{r}{\lambda_b d_b(\eta)}}\eta)>0$, and ϵ is a small positive constant close to 0. In this case, the dealer will hold at most one inventory. Then, we analyze the sign of the expected profit of each inventory as follows: $\mathbb{E}[e^{-r\tau}p_b(1)-p_s(0)]\ge e^{-\frac{r}{\lambda_b d_b(\eta)}}\eta-\left(e^{-\frac{r}{\lambda_b d_b(\eta)}}\eta-\epsilon\right)=\epsilon>0$, where τ is the random time the dealer holds the inventory, and the inequality is due to Jensen's inequality. Therefore, the dealer earns more profits than the platform in these cases, i.e., $V_{\mathcal{D}}(0)>0=V_{\mathcal{M}}$.

By (8), as κ decreases, the resulting $\boldsymbol{f}$ is the same as that after scaling up λ_b and λ_s proportionally. When $\kappa\to 0$, by a similar proof to Proposition 4 and Proposition EC.2 in the online supplement, we have $\lim_{\kappa\to 0}\sum_{m:|f_m-\hat{f}|>\epsilon}\pi_m=0$ and hence $\lim_{\kappa\to 0}V_{\mathcal{M}}=\frac{\lambda_s d_s(\hat{f})w}{r}$, which is the optimal value of problem (12) after optimizing w. From Proposition EC.1 in the online supplement, $V_{\mathcal{D}}(0)$ is upper bounded by $\bar{V}_{\mathcal{D}}(0)$, the optimal value of problem (6). In Theorem 3, we will show that the optimal values in problems (6) and (12) are equivalent. Therefore, $V_{\mathcal{D}}(0)<\lim_{\kappa\to 0}V_{\mathcal{M}}^*$. That is, the platform earns more profits than the dealer. $\square$

Proof of Theorem 3: The convergences to the static optimization problems (6) and (12) are established in Section 2.1 and 2.2. Therefore, we only need to compare (6) and (12) in the theorem. It is easy to see that the decision variables of (6), p_b and p_s, can be transformed to those of (12),

$c_s + c_b$ by the following equations: $\hat{f} = p_s, \quad \hat{f} + c_s + c_b = p_b$. There is a one-to-one correspondence between (6) and (12). Part (1) and part (2) immediately follow from this observation. Part (3) and part (4) follow from the additional observation that the average utility of sellers is $\hat{f}$ and the average utility of buyers is $\hat{f} + w$, from Proposition 4. □

EC.5. Case Studies and Economic Insights

We shall present some case studies to illustrate economic insights.[14]

Second-hand Luxury Goods Market. According to a market survey,[15] the market share of second-hand luxury goods is well above 20% in developed countries such as Japan, U.S., and France. In China, the number is only 2% but growing rapidly in the last few years. The different stages of the market also lead to different business models. In developed countries, consumers have seen the coexistence of dealership and marketplace. Designer Exchange, Resellfridges, Cocoon, BrandOff and Komehyo operate as dealers: They buy high-end luxury goods from consumers and resell it, controlling the price and inventory on their own. ThredUp, Lampoo, and Luxury Closet operate as marketplaces: They allow customers to list their used luxury products on the websites and to be directly matched with buyers. In contrast, in the Chinese market, the dealership model is dominant, including MilanStation and PonhuLuxury.

This is consistent with our theory: Theorem 2 predicts that when the market size is small, especially that of buyers, then the dealership model is more profitable, as is the case in China. Theorem 3 predicts that when the market is thick, then both models are equivalent and may coexist, as is the case in developed countries. In fact, the CEO of PonhuLuxury pointed out explicitly in an interview[16] that in the current Chinese market, a marketplace is hard to survive because of the difficulty to attract customers and make the market thick.

Used-Car Market. The used-car market is a major component of the automobile industry. It is estimated that more than 42 million used cars were sold in 2014 (Manheim 2015), and 2/3 of them were through dealers. The marketplace of used cars is growing, but eclipsed by the market share of dealership. Besides the concern for quality and information asymmetry, which is not captured by our model, Theorem 2 explains the dominance of dealership in the used-car market from the angle

[14] Our theoretical results are also consistent with empirical findings in the finance literature, although the boundary between dealership and marketplace in financial markets is blurred and the limit order books in financial markets are in general two-sided. Mayhew (2002) analyzes the data of equity option prices from 1986 to 1997, and find that dealers such as designated primary market maker works in thin markets, while traditional open outcry platform works well for liquid markets Mayhew (2002) show that for high-volume options the difference between the two market structures becomes slighter. This is consistent with our finding (Theorem 3), as for high-volume options, the market is thick and the two models are fundamentally the same.

[15] http://baogao.chinabaogao.com/baihuo/392626392626.html

[16] https://36kr.com/p/5112915

of waiting cost. The unit prices of cars are usually substantially higher than other second-hand products. This can be translated to a higher waiting cost for sellers. From Theorem 2, the dealership model is more profitable when consumers are impatient. For instance, haoche51, a Chinese online platform for used cars, started in 2014 as a marketplace to match sellers and buyers directly, and later transformed into a model that is more similar to dealership in 2016. According to an interview[17] of the CEO, the marketplace model fails because "the C2C model is not efficient ... it takes a long time for the buyers and sellers to find a match". Their new dealership is believed to better accommodate the needs of consumers.

P2P Lending Markets. In money markets, we can view lenders as sellers and borrowers as buyers. Traditional banking attracts deposits and makes loans to individuals or small and medium enterprises (SMEs). This can be viewed as a dealership model, as a bank manages its own "inventory" and makes profits from the spread. Over the past decade, technology has revolutionized the financial-service sectors. Thanks to the emergence of efficient online communication technology and data and tools available for credit modeling, P2P lending platforms, in which individual borrowers (or owners of SMEs) are matched with individual investors, have become popular since 2005, when the UK-based Zopa opened its doors. The industry has grown to $3.5bn, and continues to gain attraction.[18] These platforms are designed as a marketplace model. For example, LendingClub,[19] a US-based P2P lending company, allows borrowers to post their loan needs on the website with a target rate they would pay, and then individual lenders can browse the loan listings and select one to invest in; thus, the roles of buyers/sellers are flipped, as the buyers/borrowers are placing orders, as opposed to sellers on traditional platforms. The recent emergence of this business model is consistent with the numerical study in Table 3, in which we show that the marketplace model becomes relatively profitable for a larger market size. Indeed, the rapid increase of the consumer base is crucial for P2P lending platforms, which is only made possible by recent technology. Interestingly, when the consumer base becomes saturated, we find some companies turning to a hybrid model, consistent with our thick market analysis. Indeed, after eight years since its foundation, LendingClub began partnering with banks to enable them to purchase loans directly through the LendingClub platform or offer LendingClub products to their customers.[20]

[17] https://m.qctt.cn/news/120581

[18] https://igniteoutsourcing.com/fintech/peer-to-peer-lending-industry-trends/

[19] LendingClub is the largest P2P lending platform with a 46% market share in 2014 according to https://www.statista.com/statistics/468469/market-share-of-lending-companies-by-loans/. The second largest has 12%. LendingClub is largely a market monopoly.

[20] https://www.lendingclub.com/investing/institutional/banks

Additional Materials

EC.6. Omitted Details in Section EC.3.1.1

Algorithm 1 Algorithm to find a solution to (8) when $f^* \geq f^{**}$.

INPUT: $a \in [0, f^{**}]$
Set $n \leftarrow 1$, $X_1 \leftarrow a$ and $X_0 \leftarrow X_1 - \frac{\kappa}{\lambda_b d_b(X_1+w)}$
while True **do**
  Set $R_n \leftarrow \max_x R(x; X_n, X_{n-1})$
  if $R_n > 0$ **then**
    Solve $X_{n+1} > 0$ such that $R(X_{n+1}; X_n, X_{n-1}) = 0$ and $\frac{dR(x;X_n,X_{n-1})}{dx}|_{x=X_{n+1}} > 0$
    $n \leftarrow n+1$
  else
    Break;
  end if
end while
Set $N \leftarrow n$;
return R_N, N, $[X_N, \cdots, X_0]$.

EC.7. Additional Results for Section 2.1

We first introduce a lemma that is used for the results in Section 2.1 about the dealership market.

LEMMA EC.3. *For any $z > 0$, $\mathcal{R}_b'(z) < 0$, $\mathcal{R}_s'(z) > 0$. The functions $P_b(z)$ and $P_s(z)$ are increasing in z.*

The proof of the lemma can be found in Gallego and Topaloglu (2019).

PROPOSITION EC.3. *The value function $V_{\mathcal{D}}(x)$ satisfies the following properties:*

1. *$V_{\mathcal{D}}(x)$ is increasing and concave in x.*
2. *$V_{\mathcal{D}}(x)$ decreases in r and increases in λ_b or λ_s.*
3. *$\Delta V_{\mathcal{D}}(x)$ decreases if r increases or λ_b and λ_s decrease in proportion.*

Proof of Proposition EC.3: For part one, note that any pricing policy for initial inventory x is feasible for $x+1$. Therefore, the latter achieves a larger revenue than the former and $V_{\mathcal{D}}(x+1) > V_{\mathcal{D}}(x)$. The concavity of $V_{\mathcal{D}}(x)$ is equivalent to show

$$2V_{\mathcal{D}}(x) \geq V_{\mathcal{D}}(x+1) + V_{\mathcal{D}}(x-1), \quad x \geq 1. \tag{EC.7}$$

We use a sample-path coupling argument to prove the claim. Considering two systems, system one starts with $x+1$ units of inventory, and system two starts with x units. Assume that system one follows its optimal policies. For system two, we use the optimal policy designed for system one (which is suboptimal for system two), and couple the Poisson processes with those of system one. The coupling guarantees that the difference of inventory between two systems is always one, for all sample paths of the arrival processes before the inventory of system two reaches $x-1$. We break the coupling when the inventories of two systems drop to x and $x-1$, respectively and then let each follow its optimal policy. During coupling, the realized discounted profits of the two systems are equal. Therefore, the revenue difference between the two systems is $d(V_{\mathcal{D}}(x)-V_{\mathcal{D}}(x-1))$, where $d<1$ is the expected discounted factor at the moment coupling is broken and $V_{\mathcal{D}}(x)-V_{\mathcal{D}}(x-1)$ is the revenue difference after breaking the coupling. Since system two follows suboptimal policies for a period, we can conclude that $V_{\mathcal{D}}(x+1)-V_{\mathcal{D}}(x)$ is not larger than this difference, namely $V_{\mathcal{D}}(x+1)-V_{\mathcal{D}}(x)\le d(V_{\mathcal{D}}(x)-V_{\mathcal{D}}(x-1))\le V_{\mathcal{D}}(x)-V_{\mathcal{D}}(x-1)$. Hence (EC.7) is proved.

For part two, we again use the sample path coupling argument. We first analyze the arrival rate λ_b. Consider system A and B with arrival λ_b^A and λ_b^B ($\lambda_b^A=\lambda_b^B+\epsilon$). For system A, we implement its optimal policy $p_s^{A,*}(\cdot)$ and $p_b^{A,*}(\cdot)$. For system B, we implement a policy with $p_s^B(x)=p_s^{A,*}(x)$ and $p_b^B(\cdot)$ that satisfies $\lambda_b^A d_b(p_b^{A,*}(x))=\lambda_b^B d_b(p_b^B(x))$. Since the net arrival rates of customers and buyers are the same, we couple the net arrival processes (i.e., the thinning Poisson process with rate $p_b^A(x)d(p_b^A(x))$) of these two systems. Then, since customers are charged a higher price in system B, the revenue of system B is higher than that of system A. Therefore, $V_{\mathcal{D}}(x)$ increases in λ_b. Similarly, we can prove that $V_{\mathcal{D}}(x)$ increases in λ_s. Since the increment of the discount factor r is equivalent to the reduction of arrival rates (λ_b and λ_s) proportionally, $V_{\mathcal{D}}(x)$ decreases in the discount factor r.

For part three, note that if we scale r, λ_b and λ_s by the same constant, then $V_{\mathcal{D}}(x)$ remains unchanged from (1). So we just need to show the monotonicity of $\Delta V_{\mathcal{D}}(x)$ in r by showing the following three claims step by step.

Step (1) For all $x>0$, $\frac{d[rV_{\mathcal{D}}(x)]}{dr}\ge 0$ and $\frac{d\Delta V_{\mathcal{D}}(x)}{dr}\ge 0$ cannot hold simultaneously.

Suppose for some x, the claim above is not true. Then from (1),

$$\frac{d[rV_{\mathcal{D}}(x)]}{dr}=\lambda_b\mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))]\frac{d\Delta V_{\mathcal{D}}(x)}{dr}+\lambda_s\mathcal{R}_s'[\Delta V_{\mathcal{D}}(x+1)]\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}.$$

As $\frac{d[rV_{\mathcal{D}}(x)]}{dr}\ge 0$, $\frac{d\Delta V_{\mathcal{D}}(x)}{dr}\ge 0$, $\mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))]<0$ and $R_s'[\Delta V_{\mathcal{D}}(x+1))]>0$, we must have $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}\ge 0$. Moreover, $\frac{d[r\Delta V_{\mathcal{D}}(x+1)]}{dr}=\Delta V_{\mathcal{D}}(x+1)+r\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}$, the inequality above implies $\frac{d[rV_{\mathcal{D}}(x+1)]}{dr}>\frac{d[rV_{\mathcal{D}}(x)]}{dr}\ge 0$.

By induction, $\frac{d[rV_{\mathcal{D}}(y)]}{dr}>\frac{d[rV_{\mathcal{D}}(x)]}{dr}\ge 0$ and $\frac{d\Delta V_{\mathcal{D}}(y)}{dr}\ge 0$ for all $y>x$. However, $\frac{d\lim_{y\to\infty}[rV_{\mathcal{D}}(y)]}{dr}=\frac{d\mathcal{R}_b(0)}{dr}=0$, leading to a contradiction.

Step (2) $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$. We prove this claim by contradiction. Suppose $\frac{d[rV(0)]}{dr} \geq 0$, then from (1),

$$\frac{d[rV_{\mathcal{D}}(0)]}{dr} = \lambda_s R_s'[\Delta V_{\mathcal{D}}(1)]\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0,$$

which implies $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0$. As a result, $\frac{d[r\Delta V_{\mathcal{D}}(1)]}{dr} = \Delta V_{\mathcal{D}}(1) + r\frac{d\Delta V_{\mathcal{D}}(1)}{dr} > 0$, leading to

$$\frac{d[rV_{\mathcal{D}}(1)]}{dr} > \frac{d[rV_{\mathcal{D}}(0)]}{dr} \geq 0.$$

However, from the first step, we know $\frac{d[rV_{\mathcal{D}}(1)]}{dr} > 0$ and $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} \geq 0$ cannot hold simultaneously. Therefore, we must have $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$.

Step (3) $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$. Note that by (1), we have $\frac{d[rV_{\mathcal{D}}(0)]}{dr} = \lambda_s \mathcal{R}_s'[\Delta V_{\mathcal{D}}(1)]\frac{d\Delta V_{\mathcal{D}}(1)}{dr}$. Because $\frac{d[rV_{\mathcal{D}}(0)]}{dr} < 0$ shown in the second step, we have $\frac{d\Delta V_{\mathcal{D}}(1)}{dr} < 0$. Next, we prove the claim by induction: suppose that for some $x \geq 1$, $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$, then we must have $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} < 0$. We prove it by contradiction. Suppose $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ but $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} \geq 0$, then

$$\frac{d[r\Delta V_{\mathcal{D}}(x+1)]}{dr} = \Delta V_{\mathcal{D}}(x+1) + r\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} > 0,$$

which implies that

$$\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} > \frac{d[rV_{\mathcal{D}}(x)]}{dr} = \lambda_b \mathcal{R}_b'[\Delta V_{\mathcal{D}}(x))]\frac{d\Delta V_{\mathcal{D}}(x)}{dr} + \lambda_s \mathcal{R}_s'[\Delta V_{\mathcal{D}}(x+1)]\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr}$$

by (1). As $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ and $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} \geq 0$, then combined with the facts that $\mathcal{R}_b' < 0$ and $\mathcal{R}_s' > 0$ by Lemma EC.3, we have the last expression above being nonnegative and thus $\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} > 0$. However, $\frac{d[rV_{\mathcal{D}}(x+1)]}{dr} \geq 0$ and $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} > 0$ contradict step (1). Therefore $\frac{d\Delta V_{\mathcal{D}}(x+1)}{dr} < 0$. Therefore, by induction, we have $\frac{d\Delta V_{\mathcal{D}}(x)}{dr} < 0$ for any x. $\square$

EC.8. Additional Results for Section 2.2

We first show the following lemma.

LEMMA EC.4. *Suppose that $\tilde{x}_1 \geq y_1$ satisfying $R(\tilde{x}_1; y_1, z_1) = 0$ and $\frac{dR(x;y_1,z_1)}{dx}|_{x=\tilde{x}_1} > 0$, and $\tilde{x}_2 \geq y_2$ satisfying $R(\tilde{x}_2; y_2, z_2) = 0$. If $y_2 - z_2 \geq y_1 - z_1 \geq 0$ and $y_2 - y_1 \geq z_2 - z_1 > 0$, then $\tilde{x}_2 - \tilde{x}_1 > y_2 - y_1 > 0$.*

Proof of Lemma EC.4: From the conditions, we have

$$\begin{aligned}
0 &= R(\tilde{x}_1; y_1, z_1) - R(\tilde{x}_2; y_2, z_2) \\
&= \lambda_b[d_b(\tilde{x}_1 + w)(\tilde{x}_1 - y_1) - d_b(\tilde{x}_2 + w)(\tilde{x}_2 - y_2)] + \lambda_s[d_s(z_2)(y_2 - z_2) - d_s(z_1)(y_1 - z_1)].
\end{aligned}$$

If $y_2 - z_2 \geq y_1 - z_1 \geq 0$ and $y_2 - y_1 \geq z_2 - z_1 \geq 0$, then the last term in the last expression above is nonnegative. Therefore, the first term is nonpositive

$$d_b(\tilde{x}_1 + w)(\tilde{x}_1 - y_1) \leq d_b(\tilde{x}_2 + w)(\tilde{x}_2 - y_2). \tag{EC.8}$$

Since $y_2 > y_1$, the inequality above implies

$$d_b(\tilde{x}_1 + w)(\tilde{x}_1 - y_1) < d_b(\tilde{x}_2 + w)(\tilde{x}_2 - y_1).$$

By Assumption 1, $d_b(x+w)(x-y_1)$ and $R(x; y_1, z_1)$ are unimodal in x. Since $\frac{dR(x;y_1,z_1)}{dx}|_{x=\tilde{x}_1} > 0$, $\tilde{x}_1$ is on the left side of the mode. Therefore, the above inequality implies $\tilde{x}_1 < \tilde{x}_2$. As a result, $d_b(\tilde{x}_1 + w) > d_b(\tilde{x}_2 + w)$ and thus the inequality in (EC.8) will lead to $\tilde{x}_1 - y_1 < \tilde{x}_2 - y_2$. That is $\tilde{x}_2 - \tilde{x}_1 > y_2 - y_1 > 0$. $\square$

LEMMA EC.5. $\lim_{n\to\infty} M^{(n)} = +\infty$.

Proof of Lemma EC.5: As n increases, from the definition of f^{**} in Section EC.3.1, it satisfies $n\lambda_b d_b(f^{**} + w) f^{**} = \kappa$. Clearly, $\lim_{n\to\infty} f^{**} = 0$. By the fact that $f_{M^{(n)}} \leq f^{**}$ derived in part (3) in Lemma EC.2, we must have $\lim_{n\to\infty} f_{M^{(n)}} = 0$. From the $M^{(n)}$th equation of (8) , $\lim_{n\to\infty} d_b(f_{M^{(n)}-1} + w) f_{M^{(n)}-1} = 0$. It can only happen when $\lim_{n\to\infty} f_{M^{(n)}-1} = 0$ or $\lim_{n\to\infty} f_{M^{(n)}-1} = +\infty$. Because of part (2) in Lemma EC.2 and the unimodality, we have $\lim_{n\to\infty} f_{M^{(n)}-1} = 0$. By induction, for any finite k, $\lim_{n\to\infty} f_{M^{(n)}-k} = 0$. Suppose $M^{(n)}$ is finite, then $\lim_{n\to\infty} f_1 = 0$. That is, there will be no sellers joining the marketplace even if there are many buyers, which cannot be true. $\square$

EC.9. Algorithm for the Marketplace Model

In this section, we provide an efficient algorithm to solve the equilibrium for the marketplace model. The key is to solve f_m from the system of equations in (8). We set initial values for $f_0, f_1, \ldots$ and substitute them into the right-hand side of (8). The left-hand side is then used to update the values. Since the number of equations M is endogeneous, we set a large value $N \gg M$ and update $(f_0, \ldots, f_{N-1})$ simultaneously. The capacity M is then chosen based on the minimum m such that $f_m \leq 0$. The algorithm can be found in Algorithm 2.

Algorithm 2 Compute f_m and the expected revenue for the platform $\mathcal{R}(w)$

Require: λ_b, λ_s, $d_b(\cdot)$, $d_s(\cdot)$ and κ;

1: **function** PLATFORMREVENUE(w)

2: Set $tol \leftarrow 10^{-6}$, $error \leftarrow tol$, $N \leftarrow 2000$ and vector $\boldsymbol{f} \leftarrow \mathbf{1}$ of length N

3: **while** $error \geq tol$ **do**

4: Set vector $\boldsymbol{f}_{new} \leftarrow \mathbf{0}$ of length N

5:

$$\boldsymbol{f}_{new}(0) \leftarrow \arg\max_{f_0 \geq 0} \frac{\lambda_b d_b(f_0 + w) f_0 + \lambda_s d_s(f_2) f_2 - \kappa}{\lambda_b d_b(f_0 + w) + \lambda_s d_s(f_2)};$$

6: **for** $i = 1, 2, \ldots N-1$ **do**

$$\boldsymbol{f}_{new}(i) \leftarrow \frac{\lambda_b d_b(f_{i-1}+w) f_{i-1} + \lambda_s d_s(f_{i+1}) f_{i+1} - \kappa}{\lambda_b d_b(f_{i-1}+w) + \lambda_s d_s(f_{i+1})};$$

end for

$$\boldsymbol{f}_{new}(N) \leftarrow \frac{\lambda_b d_b(f_{N-1}+w) f_{N-1} - \kappa}{\lambda_b d_b(f_{i-1}+w)};$$

$error \leftarrow \|\boldsymbol{f}_{new} - \boldsymbol{f}\|_\infty$, $\boldsymbol{f} \leftarrow \boldsymbol{f}_{new}$;

end while

Set $M \leftarrow \arg\max_{n\geq 0} \mathbf{1} f_n \geq 0$;

for $i = 0, 1, \ldots M$ **do**

$$\pi(i) \leftarrow \frac{\prod_{j=1}^{i} \frac{\lambda_s d_s(f_j)}{\lambda_b d_b(f_{j-1}+w)}}{\sum_{k=0}^{M} \prod_{l=1}^{k} \frac{\lambda_s d_s(f_l)}{\lambda_b d_b(f_{l-1}+w)}};$$

end for

Compute the expected discounted revenue as

$$V_{\mathcal{M}}(w) \leftarrow \frac{1}{r} \sum_{i=0}^{M-1} \pi(i) \lambda_s d_s(f_{i+1}) w$$

return $V_{\mathcal{M}}(w)$, $\boldsymbol{f}$, M.

end function

EC.10. Omitted Formulation in Section 5.2

In Section 5.2, we have omitted the HJB formulation due to the page limit. We provide it here for completeness.

$$f^1_{1,0} = \max_{f^1_{0,0}\geq 0} \frac{\lambda_b d_{b,1}(f^1_{0,0}+w_1,\infty) f^1_{0,0} + \lambda_{s,1} d_{s,1}(f^1_{2,0}) f^1_{2,0} + \lambda_{s,2} d_{s,2}(f^2_{1,1}) f^1_{1,1} - \kappa}{\lambda_b d_{b,1}(f^1_{0,0}+w_1,\infty) + \lambda_{s,1} d_{s,1}(f^1_{2,0}) + \lambda_{s,2} d_{s,2}(f^2_{1,1})}$$

$$f^2_{0,1} = \max_{f^2_{0,0}\geq 0} \frac{\lambda_b d_{b,2}(\infty, f^2_{0,0}+w_2) f^2_{0,0} + \lambda_{s,1} d_{s,1}(f^1_{1,1}) f^2_{1,1} + \lambda_{s,2} d_{s,2}(f^2_{0,2}) f^2_{0,2} - \kappa}{\lambda_b d_{b,2}(\infty, f^2_{0,0}+w_2) + \lambda_{s,1} d_{s,1}(f^1_{1,1}) + \lambda_{s,2} d_{s,2}(f^2_{0,2})}$$

$$f^2_{0,x_2} = \frac{\lambda_b d_{b,2}(\infty, f^2_{0,x_2-1}+w_2) f^i_{0,x_2-1} + \lambda_{s,1} d_{s,1}(f^1_{1,x_2}) f^i_{1,x_2} + \lambda_{s,2} d_{s,2}(f^2_{0,x_2+1}) f^i_{0,x_2+1} - \kappa}{\lambda_b d_{b,2}(\infty, f^2_{0,x_2-1}+w_2) + \lambda_{s,1} d_{s,1}(f^1_{1,x_2}) + \lambda_{s,2} d_{s,2}(f^2_{0,x_2+1})} \quad x_2 \geq 2$$

$$f^1_{x_1,0} = \frac{\lambda_b d_{b,1}(f^1_{x_1-1,0}+w_1,\infty) f^i_{x_1-1,0} + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,0}) f^i_{x_1+1,0} + \lambda_{s,2} d_{s,2}(f^2_{x_1,1}) f^i_{x_1,1} - \kappa}{\lambda_b d_{b,1}(f^1_{x_1-1,0}+w_1,\infty) + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,0}) + \lambda_{s,2} d_{s,2}(f^2_{x_1,1})} \quad x_1 \geq 2$$

$$f^1_{1,x_2} = \max_{f^1_{0,x_2}\geq 0} \frac{\begin{array}{r}\lambda_b d_{b,1}(f^1_{0,x_2}+w_1, f^2_{1,x_2-1}+w_2) f^1_{0,x_2} + \lambda_b d_{b,2}(f^1_{0,x_2}+w_1, f^2_{1,x_2-1}+w_2) f^1_{1,x_2-1} \\ + \lambda_{s,1} d_{s,1}(f^1_{2,x_2}) f^1_{2,x_2} + \lambda_{s,2} d_{s,2}(f^2_{1,x_2+1}) f^1_{1,x_2+1} - \kappa\end{array}}{\begin{array}{r}\lambda_b d_{b,1}(f^1_{0,x_2}+w_1, f^2_{1,x_2-1}+w_2) + \lambda_b d_{b,2}(f^1_{0,x_2}+w_1, f^2_{1,x_2-1}+w_2) \\ + \lambda_{s,1} d_{s,1}(f^1_{2,x_2}) + \lambda_{s,2} d_{s,2}(f^2_{1,x_2+1})\end{array}} \quad x_2 \geq 1$$

$$f^2_{x_1,1} = \max_{f^2_{x_1,0}\geq 0} \frac{\begin{array}{r}\lambda_b d_{b,1}(f^1_{x_1-1,1}+w_1, f^2_{x_1,0}+w_2) f^2_{x_1-1,1} + \lambda_b d_{b,2}(f^1_{x_1-1,1}+w_1, f^2_{x_1,0}+w_2) f^2_{x_1,0} \\ + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,1}) f^2_{x_1+1,1} + \lambda_{s,2} d_{s,2}(f^2_{x_1,2}) f^2_{x_1,2} - \kappa\end{array}}{\begin{array}{r}\lambda_b d_{b,1}(f^1_{x_1-1,1}+w_1, f^2_{x_1,0}+w_2) + \lambda_b d_{b,2}(f^1_{x_1-1,1}+w_1, f^2_{x_1,0}+w_2) \\ + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,1}) + \lambda_{s,2} d_{s,2}(f^2_{x_1,2})\end{array}} \quad x_1 \geq 1$$

$$f^i_{x_1,x_2} = \frac{\begin{array}{r}\lambda_b d_{b,1}(f^1_{x_1-1,x_2}+w_1, f^2_{x_1,x_2-1}+w_2) f^i_{x_1-1,x_2} + \lambda_b d_{b,2}(f^1_{x_1-1,x_2}+w_1, f^2_{x_1,x_2-1}+w_2) f^i_{x_1,x_2-1} \\ + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,x_2}) f^i_{x_1+1,x_2} + \lambda_{s,2} d_{s,2}(f^2_{x_1,x_2+1}) f^i_{x_1,x_2+1} - \kappa\end{array}}{\begin{array}{r}\lambda_b d_{b,1}(f^1_{x_1-1,x_2}+w_1, f^2_{x_1,x_2-1}+w_2) + \lambda_b d_{b,2}(f^1_{x_1-1,x_2}+w_1, f^2_{x_1,x_2-1}+w_2) \\ + \lambda_{s,1} d_{s,1}(f^1_{x_1+1,x_2}) + \lambda_{s,2} d_{s,2}(f^2_{x_1,x_2+1})\end{array}} \quad x_1, x_2 \geq 2,$$

where $f^i_{x_1,x_2}$ is the expected utility of each type-i seller when there are x_1 type-1 sellers and x_2 type-2 sellers.

After that, we can derive the HJB equations

$$\begin{aligned}
rV_{\mathcal{M}}(0,0) &= \lambda_{s,1} d_{s,1}(f^1_{1,0}) \Delta_1 V_{\mathcal{M}}(1,0) + \lambda_{s,2} d_{s,2}(f^2_{0,1}) \Delta_2 V_{\mathcal{M}}(0,1) \\
rV_{\mathcal{M}}(0,x_2) &= \lambda_{s,1} d_{s,1}(f^1_{1,x_2}) \Delta_1 V_{\mathcal{M}}(1,x_2) + \lambda_{s,2} d_{s,2}(f^2_{0,x_2+1}) \Delta_2 V_{\mathcal{M}}(0,x_2+1) \\
&\quad + \lambda_b d_{b,2}(\infty, f^2_{0,x_2-1}+w_2)(w_2 - \Delta_2 V_{\mathcal{M}}(0,x_2)) \\
rV_{\mathcal{M}}(x_1,0) &= \lambda_{s,1} d_{s,1}(f^1_{x_1+1,0}) \Delta_1 V_{\mathcal{M}}(x_1+1,0) + \lambda_{s,2} d_{s,2}(f^2_{x_1,1}) \Delta_2 V_{\mathcal{M}}(x_1,1) \\
&\quad + \lambda_b d_{b,1}(f^1_{x_1-1,0}+w_1, \infty)(w_1 - \Delta_1 V_{\mathcal{M}}(x_1,0)) \\
rV_{\mathcal{M}}(x_1,x_2) &= \lambda_{s,1} d_{s,1}(f^1_{x_1+1,x_2}) \Delta_1 V_{\mathcal{M}}(x_1+1,x_2) + \lambda_{s,2} d_{s,2}(f^2_{x_1,x_2+1}) \Delta_2 V_{\mathcal{M}}(x_1,x_2+1) \\
&\quad + \lambda_b d_{b,1}(f^1_{x_1-1,x_2}+w_1, f^2_{x_1,x_2-1}+w_2)(w_1 - \Delta_1 V_{\mathcal{M}}(x_1,x_2)) \\
&\quad + \lambda_b d_{b,2}(f^1_{x_1-1,x_2}+w_2, f^2_{x_1,x_2-1}+w_2)(w_2 - \Delta_2 V_{\mathcal{M}}(x_1,x_2)).
\end{aligned}$$

Similar to the base model, we have

$$\begin{aligned}
rV_{\mathcal{M}} &= r\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \pi_{i,j} V_{\mathcal{M}}(i,j) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \pi_{i,j} \lambda_b \big(d_{b,1}(f^1_{i-1,j}+w_1, f^2_{i,j-1}+w_2) w_1 + d_{b,2}(f^1_{i-1,j}+w_1, f^2_{i,j-1}+w_2) w_2 \big) \\
&\quad + \sum_{i=1}^{\infty} \pi_{i,0} \lambda_b d_{b,1}(f^1_{i-1,0}+w_1, \infty) w_1 + \sum_{j=1}^{\infty} \pi_{0,j} \lambda_b d_{b,2}(\infty, f^2_{0,j-1}+w_2) w_2.
\end{aligned}$$

□

EC.11. Formulation for Multi-Product Case

In this section, we generalize the formulation in Section 5.2 to the case with n substitutable products. The arrival rate of sellers holding type-i products is $\lambda_{s,i}$ and their private WTS are denoted by $\Omega_{s,i}$. Then, the expected supply is given by $d_{s,i}(p) \triangleq \mathsf{P}(\Omega_{s,i} \leq p)$. Buyers' arrival rate is λ_b and their valuation for type-i products is $\Omega_{b,i}$. Then, given prices $\boldsymbol{p}$, the demand for type-i

product is $d_{b,i}(\boldsymbol{p}) \triangleq \mathsf{P}(\Omega_{b,i} - p_i \geq \Omega_{b,j} - p_j, \forall\, j \neq i;\; \Omega_{b,i} - p_i \geq 0)$. In the following, we use $\mathbf{0}$ to denote a vector of zeros, and $\boldsymbol{e}_i$ to denote a vector of zeros except 1 at the i-the entry.

The Dealership Market. In the dealership market, when the current inventory level is $\boldsymbol{x} \in \mathbb{N}^n$, the dealer uses the bid and ask prices $p_{b,i}(t)$ and $p_{s,i}(t)$ to maximize the expected revenue over an infinite horizon with the discount rate r. Let $\Delta_i V_{\mathcal{D}}(\boldsymbol{x}) = V_{\mathcal{D}}(\boldsymbol{x}) - V_{\mathcal{D}}(\boldsymbol{x} - \boldsymbol{e}_i)$. When $x_i = 0$, we set $\Delta_i V_{\mathcal{D}}(\boldsymbol{x}) = \infty$ such that the optimal prices satisfy $p^*_{b,i}(\boldsymbol{x}) = \infty$ for $x_i = 0$. Then, given $\boldsymbol{x} \geq 0$, the value functions satisfy the following HJB equations:

$$rV_{\mathcal{D}}(\boldsymbol{x}) = \lambda_b \max_{\boldsymbol{p}_b \geq \mathbf{0}} \Big\{ \sum_{i=1}^{n} d_{b,i}(\boldsymbol{p}_b)\big(p_{b,i} - \Delta_i V_{\mathcal{D}}(\boldsymbol{x})\big) \Big\} + \sum_{i=1}^{n} \lambda_{s,i} \max_{p_{s,i} \geq 0} d_{s,i}(p_{s,i})(\Delta_i V_{\mathcal{D}}(\boldsymbol{x} + \boldsymbol{e}_i) - p_{s,i}). \tag{EC.9}$$

The corresponding fluid approximation is

$$\begin{aligned} \bar{V}(\mathbf{0}) = \max_{\boldsymbol{p}_s, \boldsymbol{p}_b \geq \mathbf{0}} \quad & \frac{1}{r} \sum_{i=1}^{n} \lambda_{s,i} d_{s,i}(p_{s,i})(p_{b,i} - p_{s,i}) \\ s.t. \quad & \lambda_{s,j} d_{s,j}(p_{s,j}) = \lambda_b d_{b,j}(\boldsymbol{p}_b) \qquad \forall\, j = 1, 2, \ldots, n. \end{aligned}$$

When functions $d_{s,\cdot}$ and $d_{b,\cdot}$ are linear, then the optimization problem associated with the fluid approximation is convex and be solved efficiently. The solution may give a good benchmark policy for the stochastic problem, which suffers from the curse of dimensionality and is hard to solve in general.

The Marketplace. For the marketplace, we let $f^i_{\boldsymbol{x}}$ be the expected utility of a type-i seller when the current numbers of sellers are $\boldsymbol{x}$. By the same argument as the base model, we have the ask price for the type-i product as $p^i_{\boldsymbol{x}} = f^i_{\boldsymbol{x} - \boldsymbol{e}_i}$. We also let $p^i_{\boldsymbol{x}} = \infty$ when $x_i = 0$ by convention. The firm charges w_i for each transaction of type-i product. Similar to Section EC.10, we have a set of recursive equations characterizing the values of $f^i_{\boldsymbol{x}}$:

$$f^i_{\boldsymbol{x}} = \max_{f^i_{\boldsymbol{x}-\boldsymbol{e}_i} \geq 0} \frac{\sum_{j=1}^{n} \big(\mathbf{1}_{\{x_j \geq 1\}} \lambda_b d_{b,j}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1} + w_1, \cdots, f^n_{\boldsymbol{x}-\boldsymbol{e}_n} + w_n) f^i_{\boldsymbol{x}-\boldsymbol{e}_j} + \lambda_{s,j} d_{s,j}(f^j_{\boldsymbol{x}+\boldsymbol{e}_j}) f^i_{\boldsymbol{x}+\boldsymbol{e}_j}\big) - \kappa}{\sum_{j=1}^{n} \mathbf{1}_{\{x_j \geq 1\}} \lambda_b d_{b,j}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1} + w_1, \cdots, f^n_{\boldsymbol{x}-\boldsymbol{e}_n} + w_n) + \lambda_{s,j} d_{s,j}(f^j_{\boldsymbol{x}+\boldsymbol{e}_j})} \quad \text{if } x_i = 1$$

$$f^i_{\boldsymbol{x}} = \frac{\sum_{j=1}^{n} \big(\mathbf{1}_{\{x_j \geq 1\}} \lambda_b d_{b,j}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1} + w_1, \cdots, f^n_{\boldsymbol{x}-\boldsymbol{e}_n} + w_n) f^i_{\boldsymbol{x}-\boldsymbol{e}_j} + \lambda_{s,j} d_{s,j}(f^j_{\boldsymbol{x}+\boldsymbol{e}_j}) f^i_{\boldsymbol{x}+\boldsymbol{e}_j}\big) - \kappa}{\sum_{j=1}^{n} \mathbf{1}_{\{x_j \geq 1\}} \lambda_b d_{b,j}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1} + w_1, \cdots, f^n_{\boldsymbol{x}-\boldsymbol{e}_n} + w_n) + \lambda_{s,j} d_{s,j}(f^j_{\boldsymbol{x}+\boldsymbol{e}_j})} \quad \text{if } x_i \geq 2.$$

After that, we can derive the HJB equations

$$\begin{aligned} rV_{\mathcal{M}}(\boldsymbol{x}) = & \sum_{i=1}^{n} \lambda_{s,i} d_{s,i}(f^i_{\boldsymbol{x}+\boldsymbol{e}_i}) \Delta_i V_{\mathcal{M}}(\boldsymbol{x} + \boldsymbol{e}_i) \\ & + \lambda_b \sum_{i=1}^{n} \mathbf{1}_{\{x_i \geq 1\}} d_{b,i}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1} + w_1, \cdots, f^n_{\boldsymbol{x}-\boldsymbol{e}_n} + w_n)(w_i - \Delta_i V_{\mathcal{M}}(\boldsymbol{x})). \end{aligned}$$

Similar to the base model, we have

$$rV_{\mathcal{M}} = r\sum_{\boldsymbol{x}\in\mathbb{N}^n}\pi_{\boldsymbol{x}}V_{\mathcal{M}}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in\mathbb{N}^n}\sum_{i=1}^{n}\pi_{\boldsymbol{x}}\lambda_b d_{b,i}(f^1_{\boldsymbol{x}-\boldsymbol{e}_1}+w_1,\cdots,f^n_{\boldsymbol{x}-\boldsymbol{e}_n}+w_n)w_i.$$

Although we can formulate the dealership and marketplace problems for multiple products, the formulations are quite intractable. Computationally, one may resort to approximate dynamic programming to aggregate the states to lower the computational cost. Theoretically, we leave the analysis of the structural property and provable algorithmic design for future research.

EC.12. Proofs for Results in Section 5.3

Proof of Lemma 2: In the following, we study the cases when the dealer holds at most one inventory in the steady state.

Case 1: The dealer holds zero inventory in the steady state. In this case, we have the following HJB equations:

$$rV_{\mathcal{D}}(0)=0,\qquad rV_{\mathcal{D}}(1)=\lambda_b\max_{p_b\geq 0}\left\{(b_1-p_b)^+(p_b-\Delta V_{\mathcal{D}}(1))\right\}=\frac{\lambda_b}{4}(b_1-\Delta V_{\mathcal{D}}(1))^2.$$

From the above equations, we can deduce that $\Delta V_{\mathcal{D}}(1)=\frac{b_1\lambda_b+2r-2\sqrt{r(b_1\lambda_b+r)}}{\lambda_b}$. Since the dealer will not purchase from sellers, we should make sure that $\Delta V_{\mathcal{D}}(1)<b_2$, i.e., $\lambda_b<\frac{4b_2r}{(b_1-b_1)^2}$. In this case, the optimal revenue is $V_{\mathcal{D}}=0$.

Case 2: The dealer holds one inventory in the steady state. In this case, we have the following HJB equations:

$$\begin{aligned}
rV_{\mathcal{D}}(0)&=\lambda_s\max_{p_s\geq 0}\left\{\mathbf{1}_{p_s\geq b_2}(\Delta V_{\mathcal{D}}(1)-p_s)\right\}=\lambda_s(\Delta V_{\mathcal{D}}(1)-b_2)\\
rV_{\mathcal{D}}(1)&=\lambda_b\max_{p_b\geq 0}\left\{(b_1-p_b)^+(p_b-\Delta V_{\mathcal{D}}(1))\right\}=\frac{\lambda_b}{4}(b_1-\Delta V_{\mathcal{D}}(1))^2\\
rV_{\mathcal{D}}(2)&=\lambda_b\max_{p_b\geq 0}\left\{(b_1-p_b)^+(p_b-\Delta V_{\mathcal{D}}(2))\right\}=\frac{\lambda_b}{4}(b_1-\Delta V_{\mathcal{D}}(2))^2.
\end{aligned}$$

where we have $p_s^*(0)=b_2$ and $p_b^*(x)=\frac{b_1+\Delta V(x)}{2}$.

From the above HJB equations, we can deduce that

$$\begin{aligned}
\Delta V_{\mathcal{D}}(1)&=\frac{b_1\lambda_b+2(\lambda_s+r)-2\varsigma}{\lambda_b},\\
\Delta V_{\mathcal{D}}(2)&=\frac{b_1\lambda_b+2r-2\sqrt{b_1\lambda_b(\lambda_s+2r)-2(\lambda_s+r)\varsigma-b_2\lambda_b\lambda_s+2\lambda_s^2+4\lambda_s r+3r^2}}{\lambda_b},
\end{aligned}$$

where $\varsigma=\sqrt{b_1\lambda_b(\lambda_s+r)-b_2\lambda_b\lambda_s+(\lambda_s+r)^2}$.

Next, since the dealer holds at most one inventory in the steady state, we should make sure that $\Delta V_{\mathcal{D}}(1)\geq b_2\geq\Delta V_{\mathcal{D}}(2)$. In the following, we provide a sufficient condition. When $\lambda_b\geq\frac{4b_2r}{(b_1-b_2)^2}$, we have $\Delta V_{\mathcal{D}}(1) \geq b_2$. Then, it is sufficient to derive conditions when $\Delta V_{\mathcal{D}}(2) \leq b_2$, which is equivalent to the following inequality:

$$g(\lambda_b,\lambda_s,b_1,b_2) = b_1^2\lambda_b^2 - 2b_1\lambda_b(b_2\lambda_b + 2(\lambda_s+r)) + 8(\lambda_s+r)(\varsigma - \lambda_s - r) + b_2^2\lambda_b^2 + 4b_2\lambda_b(\lambda_s - r) \leq 0.$$

When $\lambda_b \in [\frac{4b_2r}{(b_1-b_2)^2}, \frac{8b_2r}{(b_1-b_2)^2}]$, we have $g(\lambda_b,\lambda_s,b_1,b_2) \leq \lambda_b((b_1-b_2)^2\lambda_b - 8b_2r) \leq 0$ due to the fact that $\varsigma = \sqrt{b_1\lambda_b(\lambda_s+r) - b_2\lambda_b\lambda_s + (\lambda_s+r)^2} \leq \lambda_s + r + \frac{\lambda_b}{2}(b_1-b_2)$. To summarize, when $\lambda_b \in [0, \frac{4b_2r}{(b_1-b_1)^2})$, the dealer holds zero inventory in the steady state and the expected profit is $V_{\mathcal{D}}^* = 0$; when $\lambda_b \in [\frac{4b_2r}{(b_1-b_1)^2}, \frac{8b_2r}{(b_1-b_1)^2})$, the dealer holds at most one inventory in the steady state and the expected profit is

$$V_{\mathcal{D}}^* = \frac{\lambda_s}{\lambda_b r(\varsigma - r)} b_2\lambda_b(-\varsigma + 2\lambda_s + r) + 2(\lambda_s+r)(\varsigma - \lambda_s - r) + b_1\lambda_b(\varsigma - 2\lambda_s - 2r). \qquad \square$$

Proof of Lemma 3: In the following, we study the cases when there is at most one seller on the market.

Case 1: No sellers join. When $\lambda_b < \frac{4\kappa}{(b_1-b_2)^2}$, we have

$$\begin{aligned} f_1 &= \max_{f_0}\Big\{\frac{\lambda_b d_b(f_0+w)f_0 + \lambda_s d_s(f_2)f_2 - \kappa}{\lambda_b d_b(f_0+w) + \lambda_s d_s(f_2)}\Big\} \leq \max_{f_0}\Big\{\frac{\lambda_b d_b(f_0+w)f_0 + \lambda_s d_s(f_2)f_1 - \kappa}{\lambda_b d_b(f_0+w) + \lambda_s d_s(f_2)}\Big\} \\ &\leq \max_{f_0}\Big\{f_0 - \frac{\kappa}{\lambda_b d_b(f_0+w)}\Big\} = b_1 - w - 2\sqrt{\frac{\kappa}{\lambda_b}} < b_2, \end{aligned}$$

which implies that no sellers will join the market and $V_{\mathcal{M}}^* = 0$.

Case 2: At most one seller on the market. When $\lambda_b \in [\frac{4\kappa}{(b_1-b_2)^2}, \frac{25\kappa}{4(b_1-b_2)^2})$, we have

$$\begin{aligned} f_1 &= \max_{f_0}\Big\{\frac{\lambda_b d_b(f_0+w)f_0 + \lambda_s d_s(f_2)f_2 - \kappa}{\lambda_b d_b(f_0+w) + \lambda_s d_s(f_2)}\Big\} \leq \max_{f_0}\Big\{\frac{\lambda_b d_b(f_0+w)f_0 + \lambda_s d_s(f_2)f_1 - \kappa}{\lambda_b d_b(f_0+w) + \lambda_s d_s(f_2)}\Big\} \\ &\leq \max_{f_0}\Big\{f_0 - \frac{\kappa}{\lambda_b d_b(f_0+w)}\Big\} = b_1 - w - 2\sqrt{\frac{\kappa}{\lambda_b}} \\ f_2 &= \frac{\lambda_b d_b(f_1+w)f_1 + \lambda_s d_s(f_3)f_3 - \kappa}{\lambda_b d_b(f_1+w) + \lambda_s d_s(f_3)} \leq \frac{\lambda_b d_b(f_1+w)f_1 + \lambda_s d_s(f_3)f_2 - \kappa}{\lambda_b d_b(f_1+w) + \lambda_s d_s(f_3)} \\ &= f_1 - \frac{\kappa}{\lambda_b d_b(f_1+w)} \leq b_1 - w - \frac{5}{2}\sqrt{\frac{\kappa}{\lambda_b}} < b_2, \end{aligned}$$

which implies that there will be at most one seller. In this case, we have

$$\begin{aligned} f_1 &= \max_{f_0}\Big\{f_0 - \frac{\kappa}{\lambda_b d_b(f_0+w)}\Big\} = b_1 - w - 2\sqrt{\frac{\kappa}{\lambda_b}}, \\ f_2 &= f_1 - \frac{\kappa}{\lambda_b d_b(f_1+w)} = b_1 - w - \frac{5}{2}\sqrt{\frac{\kappa}{\lambda_b}}, \end{aligned} \tag{EC.10}$$

where $f_0 = b_1 - w - \sqrt{\frac{\kappa}{\lambda_b}}$. Then, we can compute the value function from the following equations:

$$rV_{\mathcal{M}}(0) = \lambda_s d_s(f_1)\Delta V_{\mathcal{M}}(1), \qquad rV_{\mathcal{M}}(1) = \lambda_b d_b(f_0+w)(w - \Delta V_{\mathcal{M}}(1)).$$

From the above equations, we can deduce the expected profit in this case as $V_{\mathcal{M}}(w) = \frac{\sqrt{\kappa\lambda_b}\lambda_s w}{\sqrt{\kappa\lambda_b}r+\lambda_s r}$, where $w \le b_1 - b_2 - 2\sqrt{\frac{\kappa}{\lambda_b}}$ such that $f_1 \ge b_2$. To summarize, when $\lambda_b \in [0, \frac{4\kappa}{(b_1-b_2)^2})$, the optimal expected profit is $V^*_{\mathcal{M}} = 0$; when $\lambda_b \in [\frac{4\kappa}{(b_1-b_2)^2}, \frac{25\kappa}{4(b_1-b_2)^2})$, the optimal expected profit is $V^*_{\mathcal{M}} = \frac{\sqrt{\kappa\lambda_b}\lambda_s}{\sqrt{\kappa\lambda_b}r+\lambda_s r}(b_1 - b_2 - 2\sqrt{\frac{\kappa}{\lambda_b}})$. □

Proof of Lemma 4: In the following, we analyze the case when $\lambda_b \in [\max\{\frac{4b_2 r}{(b_1-b_2)^2}, \frac{4\kappa}{(b_1-b_2)^2}\}, \min\{\frac{8b_2 r}{(b_1-b_2)^2}, \frac{25\kappa}{4(b_1-b_2)^2}\}]$. In this case, according to Lemmas 2 and 3, both the dealership and the marketplace hold at most one inventory or seller, and the corresponding profit formulas are provided in (15) and (16). For ease of exposition, we use $\rho = \lambda_b/\lambda_s$ to denote the ratio of arrival rates.

We start with the proof of the first two parts. Given $\lambda_s + \lambda_b = \lambda$, the bid-ask spread is

$$p^*_b(1) - p^*_s(1) = b_1 - b_2 + \frac{r}{\lambda} + \frac{r+\lambda}{\lambda\rho} - \frac{1}{\lambda\rho}\sqrt{(\rho+1)^2 r^2 + \lambda(\rho+1)(2+b_1\rho)r + \lambda^2(1+b_1\rho-b_2\rho)},$$

which increases in ρ when $\lambda_b \in [\frac{4b_2 r}{(b_1-b_2)^2}, \frac{8b_2 r}{(b_1-b_2)^2}]$. The optimal transaction fee is $w^* = b_1 - b_2 - 2\sqrt{\frac{\kappa(\rho+1)}{\lambda\rho}}$, which increases in the ratio ρ.

Then, we show the change of preferred business model as ρ increases. Given $\lambda_b + \lambda_s = \lambda$ and $\rho = \lambda_b/\lambda_s$, the sign of $V^*_{\mathcal{D}} - V^*_{\mathcal{M}}$ is the same as the following function

$$\begin{aligned} h(\rho) := \sqrt{\kappa(\rho+1)\rho}\Big(&2\left(\sqrt{\kappa\lambda\rho(\rho+1)}+\lambda\right)\xi + r(\rho+1)\left(2\xi - \lambda(b_1\rho+4) - 2\sqrt{\kappa\lambda\rho(\rho+1)}\right) \\ &- 2\lambda^2(b_1\rho - b_2\rho+1) - 2r^2(\rho+1)^2\Big) + \sqrt{\lambda}\Big(r(\rho+1)\left(2\xi + \lambda(-2b_1\rho + b_2\rho - 4)\right) \\ &+ \lambda\left((b_1\rho - b_2\rho+2)\xi - 2\lambda(b_1\rho - b_2\rho+1)\right) - 2r^2(\rho+1)^2\Big), \end{aligned}$$

where $\xi = \sqrt{\lambda^2(b_1\rho - b_2\rho+1) + r\lambda(\rho+1)(b_1\rho+2) + r^2(\rho+1)^2}$.

First, we have $h(0) = 0$ and $\lim_{\rho\to 0}\frac{\partial h(\rho)}{\partial\rho} = 2\kappa\lambda^{\frac{3}{2}} > 0$, which imply that the dealership is preferred when ρ is sufficiently small. Moreover, when $\kappa < \frac{r(2r+b_1\lambda-2\sqrt{r(r+b_1\lambda)})}{4\lambda}$, we have

$$\lim_{\rho\to\infty}\frac{h(\rho)}{\rho^3} = \sqrt{\kappa}\Big(2\sqrt{r(r+\lambda b_1)\kappa\lambda} - 2r^2 - rb_1\lambda - 2r\sqrt{\kappa\lambda} + 2r\sqrt{r(r+b_1\lambda)}\Big) < 0,$$

which implies that the marketplace is better than the dealership when ρ is sufficiently large. □

References

Amihud Y, Mendelson H (1980) Dealership market: Market-making with inventory. *Journal of Financial Economics* 8(1):31–53.

Gallego G, Topaloglu H (2019) *Revenue Management and Pricing Analytics* (Springer-Verlag New York).

Manheim (2015) 2015 used car market report. Technical report, Manheim.

Mayhew S (2002) Competition, market structure, and bid-ask spreads in stock option markets. *The Journal of Finance* 57(2):931–958.